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On paranormed λ-sequence spaces of non-absolute type

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ABSTRACT

In this work, we introduce some new generalized sequence spaces related to the spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$. Furthermore, we investigate some topological properties such as the completeness and the isomorphism, and we also give some inclusion relations among these sequence spaces. In addition, we compute the α-, β- and γ-duals of these spaces, and characterize certain matrix transformations on these sequence spaces.

1. Introduction

Some basic approaches of studying the sequence spaces are inclusion relations, matrix mapping, and the determination of topologies which are completeness, duals (continuous or Köthe–Teoplitz), and bases. To obtain new sequence spaces, in general, the matrix domain μ_A of an infinite matrix A defined by $\mu_A = \{x = (x_k) \in w : Ax \in \mu\}$ is used. In most cases, the new sequence space μ_A generated by the limitation matrix A from a sequence space μ is the expansion or the contraction of the original space μ; in some cases these spaces may be overlap. Indeed, one can easily see that the inclusion $\mu_S \subset \mu$ strictly holds for $\mu \in \{\ell_\infty, c, c_0\}$. Similarly, one can deduce that the inclusion $\mu \subset \mu_\Delta$ also strictly holds for $\mu \in \{\ell_\infty, c, c_0\}$; where S and Δ are matrix operators.

Recently, in [1], Mursaleen and Noman constructed new sequence spaces by using a matrix domain over a normed space. They also studied some topological properties and inclusion relations of these spaces.

It is well known that paranormed spaces have more general properties than normed spaces. In this work, we generalize the normed sequence spaces defined by Mursaleen [1] to paranormed spaces. Furthermore, we introduce new sequence spaces over the paranormed space by using the expansion method. Then, we investigate behavior of the sequence spaces according to topological properties and inclusion relations. Finally, we give certain matrix transformation on these sequence spaces and their duals.

In the literature, the approach of constructing a new sequence space on the normed space or the paranormed space by means of the matrix domain of a particular limitation method has recently been employed by several authors; see, for example, Wang [2], Nq and Lee [3], Malkowsky and Savaş, [4] Malkowsky [5], Mursaleen and Noman [1] Altay and Başar [6,7], Altay et al. [8], Başar and Altay [9–11], Aydın and Başar [12,13], Karakaya and Polat [14], Polat et al. [15], Savaş et al. [16] and Demiriz and Çakan [17]. Some of the above-mentioned authors introduced the following sequence spaces: $(\ell_\infty)_{C_1} = X_\infty$ and $(\ell_p)_{C_1} = X_p$ in [3], $\mu_G = Z(u, v; \mu)$ in [4], $(\ell_p)_\Delta = bv_p$ and $(\ell_\infty)_\Delta = bv_\infty$ in [9], $\ell_\infty^\lambda = (\ell_\infty)_\Lambda$, $c^\lambda = (c)_\Lambda$, $c_0^\lambda = (c_0)_\Lambda$

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in [1], $\lambda(u, v; p) = \{\lambda(p)\}$ for $\lambda \in \{\ell_\infty, c, c_0\}$ in [11], $r^t_\infty(p) = (\ell_\infty(p))_{R^t}$, $r^t_c(p) = (c(p))_{R^t}$ and $r^t_0(p) = (c_0(p))_{R^t}$ in [6], and $e^r_0(\Delta; p) = (c_0(p))_{E^r\Delta}$, $e^r(\Delta; p) = (c(p))_{E^r\Delta}$ and $e^r_\infty(\Delta; p) = (\ell_\infty(p))_{E^r\Delta}$ in [14], where C_1, R^t, and E^r denote the Cesâro, the Riesz, and the Euler means, respectively, Δ denotes the band matrix of the difference operators, and Λ and G are defined in [1,4], respectively.

By w, we denote the space of all real-valued sequences. Any vector subspace of w is called a sequence space. We write ℓ_∞, c, and c_0 for the sequence space of all bounded, convergent, and null sequences, respectively. Also, by ℓ_1, cs, and bs, we denote the spaces of all absolutely convergent series, convergent series, and bounded series, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $h : X \to \mathbb{R}$ such that $h(\theta) = 0$, $h(x) = h(-x)$ and scalar multiplication is continuous; i.e., $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all α in $\mathbb{R}$ and x in X, where θ is the zero in the linear space X.

Let μ, ν be any two sequence spaces and let $A = (a_{nk})$ be any infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then, we say that A defines a matrix mapping from μ into ν by writing $A : \mu \to \nu$ if, for every sequence $x = (x_k) \in \mu$, the sequence $Ax = (A_n(x))$, the A-transform of x, is in ν, where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in N). \tag{1.1}$$

By (μ, ν), we denote the class of all matrices A such that $A : \mu \to \nu$. Thus, $A \in (\mu, \nu)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in N$ and every $x \in \mu$, and we have $Ax \in \nu$ for all $x \in \mu$. A sequence x is said to be A-summable to a if Ax converges to a, which is called as the A-limit of x.

Assume here and subsequently that (p_k), (q_k) are bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $M = \max(1, H)$. Then, the linear spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ were defined by Maddox [18] as follows.

$$\ell_\infty(p) = \{x = (x_n) \in w : \sup_{n\in N} |x_n|^{p_n} < \infty\}$$

$$c(p) = \{x = (x_n) \in w : \lim_{n\to\infty} |x_n - l|^{p_n} = 0 \text{ for some } l \in \mathbb{R}\}$$

$$c_0(p) = \{x = (x_n) \in w : \lim_{n\to\infty} |x_n|^{p_n} = 0\},$$

which are the complete spaces paranormed by

$$h(x) = \sup_{n\in N} |x_n|^{\frac{p_n}{M}}, \quad \text{iff } \inf p_n > 0.$$

Throughout this work, by $\mathcal{F}$ and $\mathbb{N}_k$, respectively, we shall denote the collection of all subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, and $e = (1, 1, 1, \ldots)$.

2. The sequence spaces $\ell_\infty(\lambda, p)$, $c(\lambda, p)$ and $c_0(\lambda, p)$

In this section, we define the sequence spaces $\mu(\lambda, p)$ for $\mu \in \{\ell_\infty, c, c_0\}$, and prove that these sequence spaces according to their paranorm are complete paranormed linear spaces. In [1], Mursaleen and Noman defined the matrix $\Lambda = (\lambda_{nk})^\infty_{n,k=0}$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n} & (0 \leq k \leq n) \\ 0 & (k > n), \end{cases} \tag{2.1}$$

where $\lambda = (\lambda_k)^\infty_{k=0}$ is a strictly increasing sequence of positive reals tending to ∞; that is, $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. Now, by using (2.1), we define new sequence spaces as follows:

$$\ell_\infty(\lambda, p) = \left\{ x = (x_k) \in w : \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^{p_n} < \infty \right\};$$

$$c(\lambda, p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1})(x_k - l) \right|^{p_n} = 0 \text{ for some } l \in \mathbb{R} \right\};$$

$$c_0(\lambda, p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^n (\lambda_k - \lambda_{k-1}) x_k \right|^{p_n} = 0 \right\}.$$

Then, we have the following special cases.

(i) If $p = e$, then $c_0(\lambda, p) = c_0^\lambda$, $c(\lambda, p) = c^\lambda$ and $\ell_\infty(\lambda, p) = \ell_\infty^\lambda$ (see [1]).

(ii) If $\lambda = (Q_n)$ and $p = e$, then $c_0(\lambda, p) = (\bar{N}, q)_0$, $c(\lambda, p) = (\bar{N}, q)$ and $\ell_\infty(\lambda, p) = (\bar{N}, q)_\infty$, where $Q_n = \sum_{k=0}^n q_k$ and $q = (q_k)$ is a positive real sequence (see [5]).

(iii) If $\lambda = (n + 1)$ and $p = e$, then $\ell_\infty(\lambda, p) = X_\infty$ (see [3]) and $\mu(\lambda, p) = [\mu]_{C_1}$ for $\mu \in \{c_0, c\}$.

For any $x = (x_n) \in w$, we define the sequence $y = (y_n)$, which will frequently be used, as the Λ-transform of x, i.e., $y = \Lambda(x)$, and hence

$$y_n = \sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) x_k \quad (n \in N). \tag{2.2}$$

We now may begin with the following theorem.

Theorem 1. *The sequence spaces $\ell_\infty(\lambda, p)$, $c(\lambda, p)$ and $c_0(\lambda, p)$ are the complete linear metric spaces with respect to paranorm defined by*

$$h(x) = \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^{\frac{p_n}{M}}.$$

Proof. We prove the theorem only for space $c_0(\lambda, p)$. For the other spaces of the theorem, a similar idea can be repeated. The linearity of $c_0(\lambda, p)$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities, which are satisfied for $x, t \in c_0(\lambda, p)$ (see [19]).

$$\sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k + t_k) \right|^{\frac{p_n}{M}} \le \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^{\frac{p_n}{M}} + \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) t_k \right|^{\frac{p_n}{M}}, \tag{2.3}$$

and, for any $\alpha \in \mathbb{R}$ (see [20]),

$$|\alpha|^{p_k} \le \max\{1, |\alpha|^M\}.$$

It is clear that $h(\theta) = 0$, $h(x) = h(-x)$ for all $x \in c_0(\lambda, p)$. Again, the inequalities (2.3) and (2.4) yield the subadditivity of h and $h(\alpha x) \le \max\{1, |\alpha|^M\} h(x)$. Let $\{x^m\}$ be any sequence of points $x^m \in c_0(\lambda, p)$ such that $h(x^m - x) \to 0$, and also let (α_m) be any sequence of scalars such that $\alpha_m \to \alpha$. Then, since the inequality

$$h(x^m) \le h(x) + h(x^m - x)$$

holds by subadditivity of h, $\{h(x^m)\}$ is bounded, and we thus have

$$\begin{aligned} h(\alpha_m x^m - \alpha x) &= \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(\alpha_m x_k^m - \alpha x_k) \right|^{\frac{p_n}{M}} \\ &\le |\alpha_m \to \alpha|^{\frac{p_n}{M}} h(x^m) + |\alpha|^{\frac{p_n}{M}} h(x^m - x), \end{aligned}$$

which tends to zero as $n \to \infty$. Therefore, the scalar multiplication is continuous. Hence, h is a paranorm on the space $c_0(\lambda, p)$. It remains to prove the completeness of the space $c_0(\lambda, p)$. Let $\{x^j\}$ be any Cauchy sequence in the space $c_0(\lambda, p)$, where $x^j = \{x_0^{(j)}, x_1^{(j)}, x_2^{(j)}, \ldots\}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $m_0(\varepsilon)$ such that $h(x^j - x^i) < \frac{\varepsilon}{2}$ for all $i, j > m_0(\varepsilon)$. Using the definition of h, we obtain for each fixed $n \in \mathbb{N}$ that

$$|\Lambda_n(x^j) - \Lambda_n(x^i)|^{\frac{p_n}{M}} \le \sup_n |\Lambda_n(x^j) - \Lambda_n(x^i)|^{\frac{p_n}{M}} < \frac{\varepsilon}{2} \tag{2.4}$$

for every $i, j > m_0(\varepsilon)$, which leads us to the fact that $\{\Lambda_n(x^0), \Lambda_n(x^1), \Lambda_n(x^2), \ldots\}$ is a Cauchy sequence of real numbers for every fixed $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, the sequence $A_n(x^i)$ convergens the sequence $A_n(x)$ as i tend to infinity. Using these infinitely many limits, we may write the sequence $\{\Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \ldots\}$. By using the (2.5) as $i \to \infty$, we have

$$|\Lambda_n(x^j) - \Lambda_n(x)| < \frac{\varepsilon}{2}, \quad (j \ge m_0(\varepsilon))$$

for every fixed $n \in \mathbb{N}$. Since $x^j = (x_k^{(j)}) \in c_0(\lambda, p)$ for each $j \in \mathbb{N}$, there exists $m_0(\varepsilon) \in \mathbb{N}$ such that $|\Lambda_n(x^j)|^{\frac{p_n}{M}} < \frac{\varepsilon}{2}$ for every $j \ge m_0(\varepsilon)$ and for each $n \in \mathbb{N}$. By taking a fixed $j \ge m_0(\varepsilon)$, we obtain by (2.5) that

$$|\Lambda_n(x)|^{\frac{p_n}{M}} \le |\Lambda_n(x^j) - \Lambda_n(x)|^{\frac{p_n}{M}} + |(\Lambda x^j)_n|^{\frac{p_n}{M}} < \varepsilon$$

for every $j \ge m_0(\varepsilon)$. Hence, we get $x \in c_0(\lambda, p)$. As a result, the space $c_0(\lambda, p)$ is complete.

Therefore, one can easily check that the absolute property does not hold on the spaces $\ell_\infty(\lambda, p)$, $c_0(\lambda, p)$ and $c(\lambda, p)$; that is, $h(x) = h(|x|)$ for at least one sequence in this space, which means that $\ell_\infty(\lambda, p)$, $c_0(\lambda, p)$ and $c(\lambda, p)$ are sequence spaces of non-absolute type, where $|x| = (|x_k|)$. □

Theorem 2. *The sequence spaces $\ell_\infty(\lambda, p)$, $c(\lambda, p)$, and $c_0(\lambda, p)$ of non-absolute type are linearly isomorphic to the spaces $\ell_\infty(p)$, $c(p)$, and $c_0(p)$, respectively, where $0 < p_k \le H < \infty$.*

Proof. We establish this for the space $c_0(\lambda, p)$. To prove the theorem, we should show the existence of a linear bijection between the spaces $c_0(\lambda, p)$ and $c_0(p)$. With the notation of (2.2), we define transformation T from $c_0(\lambda, p)$ to $c_0(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$, and hence T is injective.

Let $y \in c_0(p)$, and define the sequence $x = \{x_n\}$:

$$x_n(\lambda) = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{\lambda_k}{\lambda_n - \lambda_{n-1}} y_k \quad (n, k \in \mathbb{N}).$$

Then, we have

$$h_{c_0(\lambda,p)}(x) = \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right|^{\frac{p_n}{M}} = \sup_n |y_n|^{\frac{p_n}{M}} = h_{c_0(p)}(y).$$

Thus, we have that $x \in c_0(\lambda, p)$, and consequently T is surjective and paranorm preserving. Hence, T is a linear bijection, and this tells us that the spaces $c_0(\lambda, p)$ and $c_0(p)$ are linearly isomorphic.

It is clear here that if the spaces $c_0(\lambda, p)$ and $c_0(p)$ are respectively replaced by the spaces $c(\lambda, p), c(p)$, and $\ell_\infty(\lambda, p), \ell_\infty(p)$, then we obtain the fact that $c(\lambda, p) \cong c(p)$ and $\ell_\infty(\lambda, p) \cong \ell_\infty(p)$. This completes the proof. □

3. Some inclusion relations

In this section, we give some inclusion relations between the spaces $\ell_\infty(p), c(p), c_0(p)$ and the spaces $\ell_\infty(\lambda, p)$, $c(\lambda, p), c_0(\lambda, p)$.

Theorem 3. *The inclusions $c_0(\lambda, p) \subset c(\lambda, p) \subset \ell_\infty(\lambda, p)$ strictly hold.*

Proof. To show that the inclusion $c_0(\lambda, p) \subset c(\lambda, p)$, by using relation $y = \Lambda x$, we take $\Lambda x \in c_0(p)$. Since $c_0(p) \subset c(p)$, we get $\Lambda x \in c(p)$, and hence $x \in c(\lambda, p)$. Also, since $\Lambda x \in c(p)$ for every $x \in c(\lambda, p)$ and the inclusion $c_0(p) \subset c(p)$ is strict, for some $\Lambda x \in c(p)$, we get $\Lambda x \notin c_0(p)$. Hence, it is obtained that $x \notin c_0(\lambda, p)$. A similar idea can be used to show the inclusion $c(\lambda, p) \subset \ell_\infty(\lambda, p)$. Also, the inclusion $c(\lambda, p) \subset \ell_\infty(\lambda, p)$ is strict. For this, we take the following sequence: $x_k = (-1)^k \frac{\lambda_k + \lambda_{k-1}}{2(\lambda_k - \lambda_{k-1})}$ and $p_n = (-1)^n + \frac{2n+3}{n+1}$; we have

$$|\Lambda_n(x)|^{p_n} = \frac{1}{2^{(-1)^n + \frac{2n+3}{n+1}}}.$$

From the above equality, we get that the sequence x is in $\ell_\infty(\lambda, p)$ but not in $c(\lambda, p)$. □

Theorem 4. *If $1 \le p_n \le p_{n+1}$ for all $n \in N$, then the inclusions $c_0(p) \subset c_0(\lambda, p)$, $c(p) \subset c(\lambda, p)$ and $\ell_\infty(p) \subset \ell_\infty(\lambda, p)$ hold.*

Proof. If $p = e$, then the inclusions hold by [4, Lemmas 4.2; 4.3]. Otherwise, let $x \in c_0(p)$ be given. Then $|x|^p \in c_0$, where $|x|^p = (|x_k|^{p_k})_{k=0}^{\infty}$. Choose a fixed $n_0 \in \mathbb{N}$ such that $|x_k|^{p_k} < 1$ for all $k \ge n_0$. Then, we have for every $n > n_0$ that

$$|x_k|^{p_n} = (|x_k|^{p_k})^{p_n/p_k} \le |x_k|^{p_k}; \quad (n_0 \le k \le n), \tag{3.1}$$

since $p_k \le p_n$ for $k \le n (n \in N)$.

On the other hand, since $p = (p_n)$ is bounded, there exists a suitable constant $M > 0$ such that

$$\sum_{k=0}^{n_0-1} (\lambda_k - \lambda_{k-1}) |x_k|^{p_n} \le M \sum_{k=0}^{n_0-1} (\lambda_k - \lambda_{k-1}) |x_k|^{p_k}; \quad (n \in N). \tag{3.2}$$

Therefore, by using (3.1) and (3.2), we obtain by applying Hölder's inequality that

$$\begin{aligned}
|\Lambda_n(x)|^{p_n} &\le \left[\sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) |x_k| \right]^{p_n} \\
&\le \left[\sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) |x_k|^{p_n} \right] \left[\sum_{k=0}^{n} \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n} \right) \right]^{p_n - 1} \\
&= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) |x_k|^{p_n} \\
&= \frac{1}{\lambda_n} \left[\sum_{k=0}^{n_0-1} (\lambda_k - \lambda_{k-1}) |x_k|^{p_n} + \sum_{k=n_0}^{n} (\lambda_k - \lambda_{k-1}) |x_k|^{p_n} \right]
\end{aligned}$$

$$\leq \frac{M+1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|x_k|^{p_k}$$
$$= (M+1)\Lambda_n(|x|^p)$$

for all sufficiently large $n > n_0$.

Now, since $|x|^p \in c_0$, we have by [4, Lemma 2.1] that $\Lambda(|x|^p) \in c_0$. This leads us with the above inequality to the consequence that $\Lambda(x) \in c_0(p)$, and hence $x \in c_0(\lambda, p)$. Therefore, the inclusion $c_0(p) \subset c_0(\lambda, p)$ holds, since $x \in c_0(p)$ was arbitrary.

Also, if $x \in c(p)$, then there is $l \in \mathbb{R}$ such that $x - le \in c_0(p)$, which implies that $x - le \in c_0(\lambda, p)$, and hence $x \in c(\lambda, p)$. This shows that the inclusion $c(p) \subset c(\lambda, p)$ holds.

Similarly, with some modifications, it can be shown that the inclusion $\ell_\infty(p) \subset \ell_\infty(\lambda, p)$ holds. Thus, we leave the details to the reader. □

Theorem 5. *Let μ denote any of the spaces c_0, c and ℓ_∞.*

(i) *If $p_n > 1$ for all $n \in N$, then the inclusion $\mu^\lambda \subset \mu(\lambda, p)$ holds.*
(ii) *If $p_n < 1$ for all $n \in N$, then the inclusion $\mu(\lambda, p) \subset \mu^\lambda$ holds.*

Proof. (i) Let $x \in c_0^\lambda$. It is clear that $\Lambda(x) \in c_0$. One can find $m \in N$ such that $|\Lambda_n(x)| < 1$ for all $n \geq m$. Under condition (i), we have $|\Lambda_n(x)|^{p_n} < |\Lambda_n(x)|$ for all $n \geq m$. Hence we get $x \in c_0(\lambda, p)$.

(ii) We suppose that $x \in c_0(\lambda, p)$. Then $\Lambda(x) \in c_0(p)$, and there exists $m \in N$ such that $|\Lambda_n(x)|^{p_n} < 1$ for all $n \geq m$. To obtain the result, we consider the following inequality:

$$|\Lambda_n(x)| = (|\Lambda_n(x)|^{p_n})^{\frac{1}{p_n}} < |\Lambda_n(x)|^{p_n}$$

for all $n \geq m$. So, we get $x \in c_0^\lambda$. The proofs for the spaces c^λ, ℓ_∞^λ and $c(\lambda, p)$, $\ell_\infty(\lambda, p)$ can be repeated in a similar way. □

4. Some matrix transformations and duals of the spaces $\ell_\infty(\lambda, p)$, $c_0(\lambda, p)$ and $c(\lambda, p)$

In this section, we give the theorems determining the α-, β-, and γ- duals of the space $\mu(\lambda, p)$ for $\mu \in \{\ell_\infty, c, c_0\}$. In proving the theorems, we apply the technique used in [11]. Also, we give some matrix transformations from the spaces $\ell_\infty(\lambda, p)$, $c_0(\lambda, p)$ and $c(\lambda, p)$ into the paranormed spaces $\ell_\infty(q)$, $c_0(q)$ and $c(q)$ by using the matrix given in [1].

For the sequence space μ and ν, the set $S(\mu, \nu)$ defined by

$$S(\mu, \nu) = \{a = (a_k) \in w : ax \in \nu \text{ for all } x \in \mu\}$$

is called the multiplier spaces of μ and ν. The α-, β-, and γ-duals of a sequence space μ, which are respectively denoted by μ^α, μ^β, and μ^γ, are defined by

$$\mu^\alpha = S(\mu, \ell_1), \qquad \mu^\beta = S(\mu, cs) \qquad \mu^\gamma = S(\mu, bs).$$

We shall assume throughout this section that $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers, i.e., we drop the condition $\lambda_k \to \infty (k \to \infty)$ from the definition of λ. Then, we may begin with the following theorem, which computes the α-dual of the space $\mu(\lambda, p)$ for $\mu \in \{\ell_\infty, c, c_0\}$.

Theorem 6. *Let $\lambda = (\lambda_n)$ be a strictly increasing sequence of positive real numbers. Define the matrix $D^a = (d_{nk}^a)$ by*

$$d_{nk}^a = \begin{cases} (-1)^{n-k}\dfrac{\lambda_k}{\lambda_n-\lambda_{n-1}}a_n & (n-1 \leq k \leq n) \\ 0 & (0 \leq k \leq n-1) \text{ or } (k > n). \end{cases} \tag{4.1}$$

Then

$$\mu^\alpha(\lambda, p) = \{a = (a_n) \in w : D^a \in (\mu(p); \ell_1)\}.$$

Proof. We consider the following equality.

$$a_n x_n = \sum_{k=n-1}^{n} d_{nk}^a y_k = (D^a y)_n \quad (n \in \mathbb{N}), \tag{4.2}$$

where $D^a = (d_{nk}^a)$ is defined by (4.1).

From (4.2), it can be obtained that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \mu(\lambda, p)$ if and only if $D^a y \in \ell_1$ whenever $y \in \mu(p)$. This means that $a \in \mu^\alpha(\lambda, p)$ if and only if $D^a \in (\mu(p); \ell_1)$. Hence this completes the proof. □

The result of the theorem above corresponds to the special cases $q_n = 1$ for all $n \in \mathbb{N}$ in Theorem 5.1(1–3) (see [21]).

As a direct consequence of Theorem 6, we have the following.

Corollary 1. *Let* $K^* = \{k \in \mathbb{N} : n-1 \le k \le n\} \cap K$ *for* $K \subset \digamma$ *and* $r_k = M^{-\frac{1}{p_k}}$. *Then*

$$\ell_\infty^\alpha(\lambda, p) = \bigcap_{M\ge 1} \left\{ a = (a_n) \in w : \sup_{K^* \subset \digamma} \sum_{k\in K^*} d_{nk}^a r_k^{-1} \in \ell_1 \text{for every } M \in N_1 \right\};$$

$$c^\alpha(\lambda, p) = \bigcup_{M\ge 1} \left\{ a = (a_n) \in w : \sup_{K^* \subset \digamma} \sum_{k\in K^*} d_{nk}^a r_k^{-1} \in \ell_1 \text{and } a = (a_n) \in \ell_1 \text{for some } M \in N_1 \right\};$$

$$c_0^\alpha(\lambda, p) = \bigcup_{M\ge 1} \left\{ a = (a_n) \in w : \sup_{K^* \subset \digamma} \sum_{k\in K^*} d_{nk}^a r_k^{-1} \in \ell_1 \right\}.$$

In the following theorem, we characterize the β- and γ-duals of the space $\mu(\lambda, p)$ for $\mu \in \{\ell_\infty, c, c_0\}$.

Theorem 7. *Let* $\lambda = (\lambda_n)$ *be a strictly increasing sequence of positive real numbers, and let* $\Delta x_k = x_k - x_{k+1}$. *Define the sequence* $s^1 = (s_k^1)$, $s^2 = (s_k^2)$ *and the matrix* $B^a = (b_{nk}^a)$ *by*

$$s_k^1 = \Delta\left(\frac{a_k}{\lambda_k - \lambda_{k-1}}\right)\lambda_k, \qquad s_k^2 = \frac{a_k\lambda_k}{\lambda_k - \lambda_{k-1}}$$

$$b_{nk}^a = \begin{cases} s_k^1 & (0 \le k \le n-1) \\ s_k^2 & (k = n) \\ 0 & (k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. *Then*

$$\mu^\beta(\lambda, p) = \{a = (a_n) \in w : B^a \in (\mu(p); c)\}; \tag{4.3}$$

and

$$\mu^\gamma(\lambda, p) = \{a = (a_n) \in w : B^a \in (\mu(p); \ell_\infty)\}.$$

Proof. Consider the equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} s_k^1 y_k + s_n^2 y_n = (B^a y)_n. \tag{4.4}$$

From (4.4), it can be obtained that $ax = (a_n x_n) \in cs$ or bs whenever $x = (x_n) \in \mu(\lambda, p)$ if and only if $B^a y \in c$ or ℓ_∞ whenever $y = (y_k) \in \mu(p)$. This means that $a = (a_n) \in \mu^\beta(\lambda, p)$ or $a = (a_n) \in \mu^\gamma(\lambda, p)$ if and only if $B^a \in (\mu(p); c)$ or $B^a \in (\mu(p); \ell_\infty)$, where $\mu \in \{\ell_\infty, c, c_0\}$. Hence this completes the proof. □

From Theorem 7, we can deduce the following corollary.

Corollary 2. *Let* $r = (r_k)$ *with* $r_k = M^{-\frac{1}{p_k}}$ *for all* $k \in \mathbb{N}$. *Then*

(i) $\ell_\infty^\beta(\lambda, p) = \bigcap_{M\ge1}\{a = (a_n) \in w : s^1 r^{-1} \in \ell_1, s^2 r^{-1} \in c_0\}$;
(ii) $c^\beta(\lambda, p) = \bigcup_{M\ge1}\{a = (a_n) \in w : s^1 r \in \ell_1, s^2 r \in \ell_\infty$ *and* $s^3 \in cs\}$;
(iii) $c_0^\beta(\lambda, p) = \bigcup_{M\ge1}\{a = (a_n) \in w : s^1 r \in \ell_1, s^2 r \in \ell_\infty\}$;
(iv) $\ell_\infty^\gamma(\lambda, p) = \bigcap_{M\ge1}\{a = (a_n) \in w : s^1 r^{-1} \in \ell_1, s^2 r^{-1} \in \ell_\infty\}$;
(v) $c^\gamma(\lambda, p) = \bigcup_{M\ge1}\{a = (a_n) \in w : s^1 r \in \ell_1, s^2 r \in \ell_\infty, s^3 \in cs\}$;
(vi) $c_0^\gamma(\lambda, p) = \bigcup_{M\ge1}\{a = (a_n) \in w : s^1 r \in \ell_1, s^2 r \in \ell_\infty\}$,

where $s_k^3 = \frac{a_k\lambda_k}{\Delta(\lambda_k - \lambda_{k-1})}$.

After this step, we can give our theorems on the characterization of some matrix classes concerning the sequence space $\mu(\lambda, p)$ for $\mu \in \{c_0, c, \ell_\infty\}$.

Let $x, y \in w$ be connected by the relation $y = \Lambda(x)$. For an infinite matrix $A = (a_{nk})$, we have by using (4.4) of Theorem 7 that

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk} y_k + \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} y_m \quad (m, n \in \mathbb{N}), \tag{4.5}$$

where

$$\tilde{a}_{nk} = \left(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} - \frac{a_{n,k+1}}{\lambda_{k+1} - \lambda_k} \right) \lambda_k; \quad (n, k \in \mathbb{N}).$$

The necessary and sufficient conditions characterizing the matrix mapping between the sequence spaces $c_0(p)$, $c(p)$ and $\ell_\infty(p)$ of Maddox have been determined by Gross-Erdman [21]. Let L and M be natural numbers, and define the set K by $K = \{k \in \mathbb{N} : p_k \geq 1\}$. Before giving the theorems, let us suppose that (q_n) is a non-decreasing bounded sequence of positive real numbers, and consider the following conditions:

$$\exists M \sup_K \sum_n \left| \sum_{k \in K} \tilde{a}_{nk} M^{-\frac{1}{p_k}} \right|^{q_n} < \infty, \tag{4.6}$$

$$\sum_n \left| \sum_k \tilde{a}_{nk} \right|^{q_n} < \infty, \tag{4.7}$$

$$\forall M \sup_K \sum_n \left| \sum_{k \in K} \tilde{a}_{nk} M^{\frac{1}{p_k}} \right|^{q_n} < \infty, \tag{4.8}$$

$$\lim_n |\tilde{a}_{nk}|^{q_n} = 0 \quad (\forall k \in \mathbb{N}), \tag{4.9}$$

$$\forall L, \quad \exists M \sup_n L^{\frac{1}{q_n}} \sum_k |\tilde{a}_{nk}| M^{-\frac{1}{p_k}} < \infty, \tag{4.10}$$

$$\lim_n \left| \sum_k \tilde{a}_{nk} \right|^{q_n} = 0, \tag{4.11}$$

$$\forall M \lim_n \left(\sum_k \left| \tilde{a}_{nk} \right| M^{\frac{1}{p_k}} \right)^{q_n} = 0, \tag{4.12}$$

$$\exists \tilde{a}_k, \quad \lim_n \left| \tilde{a}_{nk} - \tilde{a}_k \right|^{q_n} = 0 \quad (\forall k \in \mathbb{N}), \tag{4.13}$$

$$\exists M \sup_n \sum_k |\tilde{a}_{nk}| M^{-\frac{1}{p_k}} < \infty, \tag{4.14}$$

$$\forall M \sup_n \sum_k |\tilde{a}_{nk}| M^{-\frac{1}{p_k}} < \infty, \tag{4.15}$$

$$\forall L, \quad \exists M \sup_n L^{\frac{1}{q_n}} \sum_k |\tilde{a}_{nk} - \tilde{a}_k| M^{-\frac{1}{p_k}} < \infty, \tag{4.16}$$

$$\exists \tilde{a}, \quad \lim_n \left| \sum_k \tilde{a}_{nk} - \tilde{a} \right|^{q_n} = 0, \tag{4.17}$$

$$\forall M \lim_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k| M^{\frac{1}{p_k}} \right)^{q_n} = 0, \tag{4.18}$$

$$\sup_n \left| \sum_k \tilde{a}_{nk} \right|^{q_n} < \infty, \tag{4.19}$$

$$\exists M, \quad \sup_n \left(\sum_k |\tilde{a}_{nk}| M^{-\frac{1}{p_k}} \right)^{q_n} < \infty, \tag{4.20}$$

$$\forall M \sup_n \left(\sum_k \left| \tilde{a}_{nk} \right| M^{\frac{1}{p_k}} \right)^{q_n} < \infty. \tag{4.21}$$

$$\left(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} \right)_{k=0}^{\infty} \in c_0(q) \quad (\forall n \in \mathbb{N}) \tag{4.22}$$

$$\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_{nk}\right)_{k=0}^{\infty}\in c(q)\quad(\forall n\in\mathbb{N}) \tag{4.23}$$

$$\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_{nk}\right)_{k=0}^{\infty}\in \ell_\infty(q)\quad(\forall n\in\mathbb{N}). \tag{4.24}$$

By using (4.3) and (4.5) and Corollary 2, we have the following results.

Theorem 8. *We have*

(i) $A\in(c_0(\lambda,p):\ell(q))$ *if and only if* (4.6) *and* (4.24) *hold.*
(ii) $A\in(c(\lambda,p):\ell(q))$ *if and only if* (4.6), (4.7) *and* (4.23) *hold.*
(iii) $A\in(\ell_\infty(\lambda,p):\ell(q))$ *if and only if* (4.8) *and* (4.22) *hold.*

Theorem 9. *We have*

(i) $A\in(c_0(\lambda,p):c_0(q))$ *if and only if* (4.9), (4.10) *and* (4.24) *hold.*
(ii) $A\in(c(\lambda,p):c_0(q))$ *if and only if* (4.9)–(4.11) *and* (4.22) *hold.*
(iii) $A\in(\ell_\infty(\lambda,p):c_0(q))$ *if and only if* (4.18), (4.13) *and* (4.22) *hold.*

Theorem 10. *We have*

(i) $A\in(c_0(\lambda,p):c(q))$ *if and only if* (4.13), (4.14), (4.16) *and* (4.23) *hold.*
(ii) $A\in(c(\lambda,p):c(q))$ *if and only if* (4.13), (4.14), (4.16), (4.17) *and* (4.23) *hold.*
(iii) $A\in(\ell_\infty(\lambda,p):c(q))$ *if and only if* (4.18), (4.21), (4.13) *and* (4.22) *hold.*

Theorem 11. *We have*

(i) $A\in(c_0(\lambda,p):\ell_\infty(q))$ *if and only if* (4.20) *and* (4.24) *hold.*
(ii) $A\in(c(\lambda,p):\ell_\infty(q))$ *if and only if* (4.19), (4.20) *and* (4.24) *hold.*
(iii) $A\in(\ell_\infty(\lambda,p):\ell_\infty(q))$ *if and only if* (4.21) *and* (4.24) *hold.*

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