

Research Article

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q -Stirling sequence spaces associated with q -Bell numbers

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Abstract: In this study, we build q -analog of the q -Stirling matrix involved q -Bell numbers $\mathbb{S}_q = (S_{nk}(q))$ defined by

$$\mathbb{S}_q = (S_{nk}(q)) = \begin{cases} \frac{S_q(n, k)}{B_q(n)}, & 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we define the sequence spaces $c(\mathbb{S}_q)$, $c_0(\mathbb{S}_q)$, $\ell_\infty(\mathbb{S}_q)$, $\ell_p(\mathbb{S}_q)$ ($1 \leq p < \infty$) using this analog. Then, we provide some inclusion relations for these spaces and examine some topological characteristics. Furthermore, we construct a basis for the space $\ell_p(\mathbb{S}_q)$, calculate α -, β -, γ -duals of the same space, and describe certain matrix classes.

Keywords: q -Stirling Numbers, q -Bell Numbers, dual spaces, matrix transform, Banach-Saks property

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1 Introduction

Stirling and Bell numbers [1] are a sequence of numbers that play an important role in combinatorics. Stirling and Bell numbers are used to represent the number of many objects. They express the number of bins in a cluster containing a certain number of elements.

q -Bell numbers are regarded as Bell numbers' q -deformations. In the same way that ordinary Bell numbers are sums of ordinary Stirling numbers, q -Bell numbers are just sums of q -Stirling numbers.

q -Stirling and q -Bell numbers are used in many mathematical fields, especially in fields such as quantum mathematics, statistical mechanics, and combinatorics, and also play an important role in quantum mechanics in topics such as matrix theory, quantum groups, and quantum field theory. For example,

q -Stirling numbers can be related to the inversion numbers of permutations, which can be used in combinatorial optimization and coding theory. q -Bell numbers can be associated with counting energy states in some quantum systems.

Several researchers have contributed to this topic. Carlitz provides important information about Stirling numbers in his notes [2]. q -Stirling numbers of the second kind and q -Bell numbers for graphs were studied by Balogh and Schlosser [3]. Generalized Stirling and q -Bell numbers (for $q = -1$) were studied by Wagner [4,5]. q -Analogues of the Catalan numbers were derived by Andrews [6]. Generalizations of Spivey's and Mező's Bell number formulas were given by Shattuck [7]. The application of q -series to various topics in analysis has been

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studied by Kac and Cheung [8]. q -Series identities and q -integral inequalities were developed by Agarwal et al. [9,10]. q -Fibonacci sequence spaces and their statistical convergence with respect to q -Fibonacci band matrix were studied by Atabey et al. [11,12]. Bell, Motzkin, and Schröder sequence spaces were investigated by Karakaş [13], Sezer et al. [14], and Dağlı [15], respectively.

Now, let us start with some notations.

q -Analog of $n \in \mathbb{N}$ is defined to be $[n]_q = n$ and

$$[n]_q = [n] = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}$$

for $q = 1$ and $q \in \mathbb{R}^+ \setminus \{1\}$, respectively.

$$[n]! = \begin{cases} [1][2]\dots[n], & n > 0, \\ 1, & n = 0 \end{cases} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

define q -factorial and q -combination, respectively. Two Pascal rules with q are defined to be

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

where $0 \leq k \leq n-1$.

Lemma 1.

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{k(n-k)} + \dots$$

is a polynomial of degree $k(n-k)$ of the variable q .

Carlitz [2, equation (3.3)] provided the explicit formula that follows:

$$S_q(n, k) = \frac{1}{[k]!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} [k-j]_q^n,$$

where $S_q(n, k) = 0$ for $k > n$ or $k < 0$ and $S_q(0, 0) = 1$. Similarly, the q -Bell numbers are defined by

$$B_q(n) = \sum_{k=0}^n S_q(n, k).$$

The q -Stirling and q -Bell numbers become the ordinary Stirling and Bell sequence of numbers when $q \rightarrow 1$.

Let the set of all sequence spaces be represented by ω . The subspaces of ω that are ℓ_∞ , c , c_0 , and ℓ_p are characterized as bounded, convergent, null, and p -absolutely summable sequence spaces, respectively. The spaces c_0 , c , ℓ_∞ are Banach spaces for $k \in \mathbb{N}$ under normed by

$$\|u\|_\infty = \sup_{k \in \mathbb{N}} |u_k|$$

and the ℓ_p ($1 \leq p < \infty$) is a Banach space normed by

$$\|u\|_p = \left(\sum_k |u_k|^p \right)^{\frac{1}{p}}.$$

Moreover, we designate the spaces of all absolutely convergent series, convergent series, bounded series, and p -bounded variation, respectively, by the notations ℓ_1 , cs , bs , and bv_p .

Let $U, V \subset \omega$ and $B = (b_{nk})$ is a real infinite matrix. The matrix B defines a matrix transformation from U to V if for every sequence $u \in U$,

$$Bu = (B_n(u)) = \left(\sum_{k=1}^{\infty} b_{nk} u_k \right) \in U$$

for each $n \in \mathbb{N}$. (U, V) represents the family of all matrices that map from U to V .

$$U_B = \{u \in \omega : Bu \in U\} \quad (1)$$

is a sequence space that defines the B 's matrix domain U_B in a sequence space U .

Moreover, the sequence space U_B is a BK -space normed by $\|u\|_{U_B} = \|Bu\|_U$ if B is a triangular matrix and U is a BK -space.

Several authors have utilized q -numbers or special matrices in summability theory, such as the q -Cesaro matrix analog was first studied by Aktuğlu and Bekar [16], and Bekar [17]. Duals and matrix transformations of the spaces c and c_0 generated with the q -Cesaro matrix analog were studied by Demiriz and Şahin [18], and those of ℓ_p and ℓ_∞ were studied by Yaying et al. [19]. The q -double Cesaro analog and its statistical convergence were studied by Çınar and Et [20]. In addition, four-dimensional matrices by Nuray and Patterson [21,22], q -Hausdorff matrices by Selmanogullari et al. [23], Triangular q -Fibonacci matrices by Atabey et al. [24], and q -Catalan and q -Pascal sequence spaces by Yaying et al. [25,26] were studied.

2 Main result

Using q -analog of the q -Stirling matrix involved q -Bell numbers for $q > 0$, we present the sequence spaces $c_0(\mathbb{S}_q)$, $c(\mathbb{S}_q)$, $\ell_\infty(\mathbb{S}_q)$, and $\ell_p(\mathbb{S}_q)$ ($1 \leq p < \infty$) in this section. After that, a Schauder basis for $\ell_p(\mathbb{S}_q)$ will be constructed, and some inclusion relations will be shown.

Given a nk th Stirling number $S_q(n, k)$ for $n, k \in \mathbb{N}$ and $q > 0$,

$$\mathbb{S}_q = (S_{nk}(q)) = \begin{cases} \frac{S_q(n, k)}{B_q(n)}, & 0 \leq k \leq n, \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{bmatrix} \frac{S_q(0,0)}{B_q(0)} & 0 & 0 & 0 & 0 & \dots \\ \frac{S_q(1,0)}{B_q(1)} & \frac{S_q(1,1)}{B_q(1)} & 0 & 0 & 0 & \dots \\ \frac{S_q(2,0)}{B_q(2)} & \frac{S_q(2,1)}{B_q(2)} & \frac{S_q(2,2)}{B_q(2)} & 0 & 0 & \dots \\ \frac{S_q(3,0)}{B_q(3)} & \frac{S_q(3,1)}{B_q(3)} & \frac{S_q(3,2)}{B_q(3)} & \frac{S_q(3,3)}{B_q(3)} & 0 & \dots \\ \frac{S_q(4,0)}{B_q(4)} & \frac{S_q(4,1)}{B_q(4)} & \frac{S_q(4,2)}{B_q(4)} & \frac{S_q(4,3)}{B_q(4)} & \frac{S_q(4,4)}{B_q(4)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

defines q -analog of the q -Stirling matrix involved q -Bell numbers.

For $n \in \mathbb{N}$, the matrix transformation $y_n = (\mathbb{S}_q)_n(x)$ is denoted by

$$y_n = \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) x_k \quad (2)$$

and the sequence spaces $c_0(\mathbb{S}_q)$, $c(\mathbb{S}_q)$, $\ell_\infty(\mathbb{S}_q)$, and $\ell_p(\mathbb{S}_q)$ ($1 \leq p < \infty$) are defined by

$$c_0(\mathbb{S}_q) = \{x = (x_n) \in \omega : \lim_{n \rightarrow \infty} (\mathbb{S}_q)_n(x) = 0\},$$

$$c(\mathbb{S}_q) = \{x = (x_n) \in \omega : \lim_{n \rightarrow \infty} (\mathbb{S}_q)_n(x) \text{ exists}\},$$

$$\ell_\infty(\mathbb{S}_q) = \left\{ x = (x_n) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) x_k \right| < \infty \right\},$$

$$\ell_p(\mathbb{S}_q) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) x_k \right|^p < \infty \right\}.$$

The sequence spaces $\ell_p(\mathbb{S}_q)$, $\ell_\infty(\mathbb{S}_q)$, $c_0(\mathbb{S}_q)$, and $c(\mathbb{S}_q)$ can be redefined by

$$\ell_p(\mathbb{S}_q) = (\ell_p)_{\mathbb{S}_q} \quad (1 \leq p < \infty), \quad \ell_\infty(\mathbb{S}_q) = (\ell_\infty)_{\mathbb{S}_q}, \quad (3)$$

$$c_0(\mathbb{S}_q) = (c_0)_{\mathbb{S}_q} \quad \text{and} \quad c(\mathbb{S}_q) = (c)_{\mathbb{S}_q}. \quad (4)$$

respectively, when (1) notation is considered.

Theorem 1. The $\ell_p(\mathbb{S}_q)$ is a BK-space normed by

$$\|(\mathbb{S}_q)_n(x)\|_{\ell_p} = \|x\|_{\ell_p(\mathbb{S}_q)} = \left(\sum_n |(\mathbb{S}_q)_n(x)|^p \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty)$$

and the $U(\mathbb{S}_q)$ are BK-spaces normed by

$$\|(\mathbb{S}_q)_n(x)\|_U = \|x\|_{U(\mathbb{S}_q)} = \sup_{n \in \mathbb{N}} |(\mathbb{S}_q)_n(x)|,$$

where $U \in \{\ell_\infty, c, c_0\}$.

Proof. The matrix $\mathbb{S}_q$ is a triangle and ℓ_∞ and ℓ_p are BK-spaces in terms of their natural norms, because (3) and (4) hold; Theorem 4.3.12 of [27, p. 63] states that the spaces $\ell_p(\mathbb{S}_q)$ and $\ell_\infty(\mathbb{S}_q)$ are BK-spaces with the given norms, where $(1 \leq p < \infty)$.

The spaces $c_0(\mathbb{S}_q)$ and $c(\mathbb{S}_q)$ are BK-spaces with the stated norms, according to [27, p. 61] Theorem 4.3.2. $\square$

Theorem 2. Suppose

$$y_n = \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) x_k,$$

for all $n \geq 0$, where $S_q(n, k)$ and $B_q(n)$ are the q -analogs of the Stirling and Bell numbers, as defined. Then we have

$$x_n = \sum_{k=0}^n s_q(n, k) B_q(k) y_k \quad n \geq 0, \quad (5)$$

where $s_q(n, k) = q^{-\binom{n}{2}} \tilde{s}_q(n, k)$ and $\tilde{s}_q(n, k)$ is the q -Stirling number of the first kind given recursively by

$$\tilde{s}_q(n, k) = \tilde{s}_q(n-1, k-1) - [n-1]_q \tilde{s}_q(n-1, k) \quad n, k \geq 1,$$

with $\tilde{s}_q(n, 0) = \delta_{n,0}$ and $\tilde{s}_q(0, k) = \delta_{0,k}$ for all $n, k \geq 0$.

Proof. Let $\tilde{S}_q(n, k)$ denote the q -Stirling number of the second kind given by

$$\tilde{S}_q(n, k) = \tilde{S}_q(n-1, k-1) - [k]_q \tilde{S}_q(n-1, k) \quad n, k \geq 1,$$

with $\tilde{S}_q(n, 0) = \delta_{n,0}$ and $\tilde{S}_q(0, k) = \delta_{0,k}$ for all $n, k \geq 0$. By [4, equation (2.3)], we have $S_q(n, k) = q^{-\binom{n}{2}} \tilde{S}_q(n, k)$ for all n and k . By the $q = 1 + t$ case of equations (9.9) and (9.10) in [28], we have the orthogonality relations

$$\sum_{k=m}^n \tilde{s}_q(n, k) \tilde{S}_q(k, m) = \sum_{k=m}^n \tilde{S}_q(n, k) \tilde{s}_q(k, m) = \delta_{n,m} \quad 0 \leq m \leq n.$$

Thus, by the definitions of the sequences, we have that $s_q(n, k)$ and $S_q(n, k)$ satisfy

$$\sum_{k=m}^n S_q(n, k) s_q(k, m) = \delta_{n,m} \quad 0 \leq m \leq n, \quad (6)$$

and hence, also $\sum_{k=m}^n s_q(n, k) S_q(k, m) = \delta_{n,m}$. To realize the second equality, note that (6) implies that the $(N+1) \times (N+1)$ matrix whose (i, j) th entry is $S_q(i, j)$ for all $0 \leq i, j \leq N$ for some N greater than or equal, the fixed n under consideration is the left inverse to the comparable $(N+1) \times (N+1)$ matrix whose (i, j) th entry is $s_q(i, j)$. Since both matrices are square, the former matrix is also the right inverse of the latter, which implies the second orthogonality relation. Thus, by the definition of the sequence y_k for $k \geq 0$, we have

$$\begin{aligned} \sum_{k=0}^n s_q(n, k) B_q(k) y_k &= \sum_{k=0}^n s_q(n, k) \sum_{j=0}^k S_q(k, j) x_j = \sum_{j=0}^n x_j \sum_{k=j}^n s_q(n, k) S_q(k, j) \\ &= \sum_{j=0}^n x_j \delta_{n,j} = x_n, \end{aligned}$$

which yields (5). $\square$

Theorem 3.

- (i) The $\ell_p(\mathbb{S}_q)$ ($1 \leq p \leq \infty$) is linearly isomorphic to the ℓ_p .
- (ii) The $c_0(\mathbb{S}_q)$ and the $c(\mathbb{S}_q)$ are linearly isomorphic to c_0 and c , respectively.

Proof. (i) To prove that $S : \ell_p(\mathbb{S}_q) \rightarrow \ell_p$, ($x \rightarrow y = Sx = \mathbb{S}_q x \in \ell_p$), is a linear and bijection transformation for $(1 \leq p \leq \infty)$ is sufficient.

S is obviously linear. In addition, S is implied to be injective as it is evident that $x = 0$ whenever $Sx = 0$.

Let us take $y = (y_n) \in \ell_p$ to show that S is surjective. We have

$$y_n = \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) x_k$$

and so

$$x_k = \sum_{i=0}^k s_q(k, i) B_q(i) y_i.$$

For $(1 \leq p < \infty)$ we consider

$$\begin{aligned} \|x\|_{\ell_p(\mathbb{S}_q)} &= \left(\sum_n |(\mathbb{S}_q)_n(x)|^p \right)^{\frac{1}{p}} = \left(\sum_n \left| \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) x_k \right|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_n \frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k) \left| \sum_{i=0}^k s_q(k, i) B_q(i) y_i \right|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_n |y_n|^p \right)^{\frac{1}{p}} = \|y\|_p < \infty \end{aligned}$$

and

$$\|x\|_{\ell_\infty(\mathbb{S}_q)} = \sup_{n \in \mathbb{N}} |(\mathbb{S}_q)_n(x)| = \|y\|_\infty < \infty.$$

The proof is now complete.

- (ii) A similar method may be used to prove the theorem using Theorem 3 (i). $\square$

Theorem 4. The inclusions $c \subset c(\mathbb{S}_q)$ and $c_0 \subset c_0(\mathbb{S}_q)$ hold for $q \geq 1$, respectively.

Proof. For any real number l and each $q \geq 1$, let us take $x \in c$ and this means that $x \rightarrow l$. The method $\mathbb{S}_q$ is regular since the matrix $\mathbb{S}_q$ satisfies the Silverman-Toeplitz criteria:

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sum_k |S_{nk}(q)| &= \sup_{n \in \mathbb{N}} \left(\sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} = \frac{S_q(n, 0) + S_q(n, 1) + \dots + S_q(n, n)}{B_q(n)} = \frac{B_q(n)}{B_q(n)} \right) = 1, \\ \lim_{n \rightarrow \infty} S_{nk}(q) &= \lim_{n \rightarrow \infty} (S_{0k}(q), S_{1k}(q), \dots) = 0 \quad \text{for each } k = 0, 1, \dots, \\ \lim_{n \rightarrow \infty} \sum_k S_{nk}(q) &= \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} \right) = 1 < \infty. \end{aligned}$$

Then, we can see that $\mathbb{S}_q x \rightarrow l$. So $x \in c(\mathbb{S}_q)$. In order to prove $c_0 \subset c_0(\mathbb{S}_q)$, $l = 0$ is necessary. $\square$

Theorem 5. The inclusion $\ell_p \subset \ell_p(\mathbb{S}_q)$ holds for $q \geq 1$, and the inclusion is strict for $q < 1$, where $1 \leq p \leq \infty$.

Proof. Proving a number $K > 0$'s existence is sufficient to demonstrate that for every $x \in \ell_p$, $\|x\|_{\ell_p(\mathbb{S}_q)} \leq K\|x\|_p$. For $(1 < p < \infty)$ and $q \geq 1$, let us take $x \in \ell_p$. Applying from Hölder's inequality for $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} |(\mathbb{S}_q)_n(x)|^p &= \sum_{n=0}^{\infty} \left| \sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} x_k \right|^p \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} |x_k| \right)^p \\ &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} |x_k|^p \right) \left(\sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} \right)^{p-1} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} |x_k|^p \right) \\ &= \sum_{k=0}^{\infty} |x_k|^p \left(\sum_{n=k}^{\infty} \frac{1}{B_q(n)} \right). \end{aligned}$$

This means

$$\|x\|_{\ell_p(\mathbb{S}_q)} \leq K\|x\|_p, \quad (7)$$

where $K = \sup_{k \in \mathbb{N}} \left(\sum_{n=k}^{\infty} \frac{S_q(n, k)}{B_q(n)} \right)$. Also, for $p = \infty$, we take $x_k \in \ell_{\infty}$. Then, for all $k \in \mathbb{N}$, there exists a constant $K > 0$ such that $|x_k| \leq K$. Therefore, using the triangle inequality

$$|(\mathbb{S}_q)_n(x)| \leq \sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} |x_k| \leq \sum_{k=0}^n \frac{S_q(n, k)}{B_q(n)} K = K.$$

So $x \in \ell_p(\mathbb{S}_q)$.

Similarly, inequality (7) is easily demonstrated for $p = 1$; hence, we omit the details. Therefore, for $1 \leq p \leq \infty$, the inclusion $\ell_p \subset \ell_p(\mathbb{S}_q)$ holds.

To show that the inclusion is strict, consider the sequence $x = (x_k) = \sum_{i=0}^k S_q(k, i) B_q(i)$; in this case, $x \in \ell_p(\mathbb{S}_q)$ but $x \notin \ell_p$. $\square$

Theorem 6.

- (i) $\ell_p(\mathbb{S}_q) \subset \ell_s(\mathbb{S}_q)$, if $1 \leq p < s$.
- (ii) For $q > 0$, the inclusion $c_0(\mathbb{S}_q) \subset c(\mathbb{S}_q)$ is strict.
- (iii) For $q > 0$, the inclusion $\ell_p(\mathbb{S}_q) \subset \ell_{\infty}(\mathbb{S}_q)$ is strict.

Proof. (i) Let us take $1 \leq p < s$ and $x \in \ell_p(\mathbb{S}_q)$. Then, we can deduce from Theorem 1 that $y \in \ell_p$ as defined in transformation (2). Consequently, utilizing the established $\ell_p \subset \ell_s$ inclusion, we can determine that $y \in \ell_s$. As a result, $x \in \ell_s(\mathbb{S}_q)$, demonstrating the validity of the inclusion $\ell_p(\mathbb{S}_q) \subset \ell_s(\mathbb{S}_q)$. This finalizes the proof.

(ii) Let us take $x \in c_0(\mathbb{S}_q)$. Then, we have $\lim_{n \rightarrow \infty} (\mathbb{S}_q)_n(x) = 0$. Since $\lim_{n \rightarrow \infty} (\mathbb{S}_q)_n(x)$ exists, we can write $x \in c(\mathbb{S}_q)$.

For strict inclusion, let us consider the sequence $x = (x_k) = (1^k)$ and calculate the transformation sequence

$$\lim_{n \rightarrow \infty} \left(\frac{1}{B_q(n)} \sum_{k=0}^n S_q(n, k)(1^k) \right) = 1 \neq 0,$$

and so this means $x \in c_0(\mathbb{S}_q) \setminus c(\mathbb{S}_q)$.

(iii) Let us take $x = (x_n) \in \ell_p(\mathbb{S}_q)$. Then, we have $\mathbb{S}_q x \in \ell_p$. Since $\ell_p \subset \ell_\infty$, we can conclude that $\mathbb{S}_q x \in \ell_\infty$. So $x = (x_n) \in \ell_\infty(\mathbb{S}_q)$, which means $\ell_p(\mathbb{S}_q) \subset \ell_\infty(\mathbb{S}_q)$. The sequence $x = (x_n) = (1^n)$ is examined for the strict inclusions. Since

$$\sup_{n \in \mathbb{N}} \left| \frac{S_q(n, k)}{B_q(n)} (1^n) \right| = 1 < \infty,$$

we have $x \in \ell_\infty(\mathbb{S}_q)$. But since

$$\sum_n \left| \frac{S_q(n, k)}{B_q(n)} (1^n) \right|^p = \sum_n |1|^p \rightarrow \infty,$$

we have $x \notin \ell_p(\mathbb{S}_q)$. □

Theorem 7. The space $\ell_p(\mathbb{S}_q)$, $p \in [1, \infty] - \{2\}$, is not a Hilbert space and is not absolute type, where $1 \leq p \leq \infty$.

Proof. We use the sequences

$$v = (v_n) = \left(\frac{B_q(0)}{S_q(0, 0)}, \frac{B_q(1)S_q(0, 0) - B_q(0)S_q(1, 0)}{S_q(0, 0)S_q(1, 1)}, \right. \\ \left. \frac{-B_q(0)S_q(2, 0)S_q(1, 1) - B_q(1)S_q(0, 0)S_q(2, 1) + B_q(0)S_q(1, 0)S_q(2, 1)}{S_q(0, 0)S_q(1, 1)S_q(2, 2)}, \dots \right)$$

and

$$u = (u_n) = \left(\frac{B_q(0)}{S_q(0, 0)}, -\frac{B_q(1)S_q(0, 0) + B_q(0)S_q(1, 0)}{S_q(0, 0)S_q(1, 1)}, \right. \\ \left. \frac{-B_q(0)S_q(2, 0)S_q(1, 1) + B_q(1)S_q(0, 0)S_q(2, 1) + B_q(0)S_q(1, 0)S_q(2, 1)}{S_q(0, 0)S_q(1, 1)S_q(2, 2)}, \dots \right)$$

for proof. These sequences contain the following $\mathbb{S}_q$ transformations:

$$\mathbb{S}_q v = (1, 1, 0, 0, \dots) \quad \text{and} \quad \mathbb{S}_q u = (1, -1, 0, 0, \dots),$$

respectively.

Thus, $\mathbb{S}_q(v + u) = (2, 0, 0, 0, \dots)$ and $\mathbb{S}_q(v - u) = (0, 2, 0, 0, \dots)$ are obtained. Hence, the expression for $p \neq 2$ that results is as follows:

$$\|v + u\|_{\ell_p(\mathbb{S}_q)}^2 + \|v - u\|_{\ell_p(\mathbb{S}_q)}^2 = 8 \neq 2^{2+\frac{2}{p}} = 2(\|v\|_{\ell_p(\mathbb{S}_q)}^2 + \|u\|_{\ell_p(\mathbb{S}_q)}^2).$$

This implies that the parallelogram equality cannot be satisfied by the norm of the space $\ell_p(\mathbb{S}_q)$.

To show that it is not an absolute type, let us take a sequence defined by $u = (1, -1, 0, 0, \dots)$. Next, we compute transformations $\mathbb{S}_q u$ and $\mathbb{S}_q |u|$ as follows:

$$\mathbb{S}_q u = \left(\frac{S_q(0, 0)}{B_q(0)}, \frac{S_q(1, 0) - S_q(1, 1)}{B_q(1)}, \frac{S_q(2, 0) - S_q(2, 1)}{B_q(2)}, 0, \dots \right)$$

and

$$\mathbb{S}_q|u| = \left(\frac{S_q(0, 0)}{B_q(0)}, \frac{S_q(1, 0) + S_q(1, 1)}{B_q(1)}, \frac{S_q(2, 0) + S_q(2, 1)}{B_q(2)}, 0, \dots \right),$$

where $|u| = |u_n|$. Since $\|u\|_{\ell_p(\mathbb{S}_q)} \neq \| |u| \|_{\ell_p(\mathbb{S}_q)}$, the proof is completed. $\square$

For $\ell_p(\mathbb{S}_q)$ ($1 \leq p < \infty$), we now provide a basis.

Theorem 8. For $1 \leq p < \infty$ and each fixed $k \in \mathbb{N}$, define a sequence $\xi^{(k)} \in \ell_p(\mathbb{S}_q)$ as

$$(\xi^{(k)})_n = \begin{cases} s_q(n, k)B_q(k), & 0 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases} \quad (k, n \in \mathbb{N}). \quad (8)$$

Later, $\{\xi^{(k)}\}_{k \in \mathbb{N}}$ is a Schauder basis for the space $\ell_p(\mathbb{S}_q)$ and each $u \in \ell_p(\mathbb{S}_q)$ has a unique representation of the form

$$u = \sum_k (\mathbb{S}_q)_k(u) \xi^{(k)} \quad (9)$$

for each $k \in \mathbb{N}$.

Proof. Let us consider $1 \leq p < \infty$. Afterward, it is clear by (8) that $(\mathbb{S}_q)(\xi^{(k)}) = e^{(k)} \in \ell_p$, and hence, $\xi^{(k)} \in \ell_p(\mathbb{S}_q)$.

Let us take $u \in \ell_p(\mathbb{S}_q)$, and for each non-negative integer m and all $k \in \mathbb{N}$, we put

$$u^{(m)} = \sum_k (\mathbb{S}_q)_k(u) \xi^{(k)}.$$

Then, we can obtain

$$\mathbb{S}_q(u^{(m)}) = \sum_{k=0}^m (\mathbb{S}_q)_k(u) (\mathbb{S}_q)(\xi^{(k)}) = \sum_{k=0}^m (\mathbb{S}_q)_k(u) e^{(k)}$$

and then

$$(\mathbb{S}_q)_n(u - u^{(m)}) = \begin{cases} 0, & (0 \leq n \leq m), \\ (\mathbb{S}_q)_n(u), & (n > m), \end{cases} \quad (n, m \in \mathbb{N}). \quad (10)$$

For any given $\varepsilon > 0$, there is a $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m_0+1}^{\infty} |(\mathbb{S}_q)_n(u)|^p = \left(\frac{\varepsilon}{2} \right)^p.$$

As a result, for every $m > m_0$, we acquire

$$\|u - u^{(m)}\|_{\ell_p(\mathbb{S}_q)} = \left(\sum_{k=m+1}^{\infty} |(\mathbb{S}_q)_n(u)|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=m_0+1}^{\infty} |(\mathbb{S}_q)_n(u)|^p \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2} < \varepsilon,$$

demonstrating that $\lim_{m \rightarrow \infty} \|u - u^{(m)}\|_{\ell_p(\mathbb{S}_q)} = 0$ and as a result, u can be stated as in (9).

To demonstrate the uniqueness of the expression, we assume the existence of another form (9), similar to

$$u = \sum_k (\mathbb{J}_q)_k(u) \xi^{(k)}.$$

By using the continuous transform S , we have proved its isomorphism in Theorem 3, and the equation that follows may be written as

$$(\mathbb{S}_q)_n(u) = \sum_k (\mathbb{J}_q)_k(u) (\mathbb{S}_q)_n(\xi^{(k)}) = \sum_k (\mathbb{J}_q)_k(u) \delta_{nk} = (\mathbb{J}_q)_n(u).$$

This proves that the form (9) is unique. This concludes the proof. $\square$

3 α -, β -, γ -duals of $\ell_p(\mathbb{S}_q)$

The α -, β -, γ -duals of $\ell_p(\mathbb{S}_q)$ are given in this section. Since $p = 1$ can be demonstrated by analogy, we will focus on the case $1 < p \leq \infty$. We serve the lemmas in Stieglitz and Tietz [29] to prove Theorems 9 and 10.

Note that $(p^{-1} + r^{-1}) = 1$ for $(1 < p \leq \infty)$ and that F represents the family of all finite subsets of $\mathbb{N}$.

Lemma 2. [29]

(i) $B = (b_{nk}) \in (\ell_p, \ell_1) \Leftrightarrow$

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} b_{nk} \right|^r < \infty.$$

(ii) $B = (b_{nk}) \in (\ell_p, c) \Leftrightarrow$

$$\text{For } (\forall k \in \mathbb{N}) \lim_{n \rightarrow \infty} b_{nk} \text{ exists} \quad (11)$$

$$\sup_{n \in \mathbb{N}} \sum_k |b_{nk}|^r < \infty. \quad (12)$$

(iii) $B = (b_{nk}) \in (\ell_\infty, c) \Leftrightarrow (11) \text{ holds and}$

$$\lim_{n \rightarrow \infty} \sum_k |b_{nk}| = \sum_k \lim_{n \rightarrow \infty} |b_{nk}|. \quad (13)$$

Lemma 3. $B = (b_{nk}) \in (\ell_p, \ell_\infty) \Leftrightarrow (12) \text{ holds with } (1 < p \leq \infty)$.

Theorem 9. The set

$$D_1(q) = \left\{ b = (b_k) \in \omega : \sup_{K \in F} \sum_k \left| \sum_{n \in K} s_q(n, k) B_q(k) b_n \right|^r < \infty \right\}$$

is the α -dual of the space $\ell_p(\mathbb{S}_q)$, where $1 < p \leq \infty$.

Proof. For $1 < p \leq \infty$ and any sequence $b = (b_n) \in \omega$, let us define a matrix G by

$$G = (g_{nk}) = \begin{cases} s_q(n, k) B_q(k) b_n, & 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for each $x = (x_n) \in \omega$, we obtain $y = \mathbb{S}_q x$. After it tracks by (2)

$$b_n x_n = \sum_{k=0}^n s_q(n, k) B_q(k) b_n y_k = G_n(y) \quad (n \in \mathbb{N}). \quad (14)$$

Because of (14), we obtain that $bx = (b_n x_n) \in \ell_1$ whenever $x \in \ell_p(\mathbb{S}_q)$ if and only if $Gy \in \ell_1$ whenever $y \in \ell_p$.

We can see from Lemma 2 (i) that

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} s_q(n, k) B_q(k) b_n \right|^r < \infty$$

and so $(\ell_p(\mathbb{S}_q))^\alpha = D_1(q)$. □

Theorem 10. Define the following sets $D_2(q)$, $D_3(q)$, $D_4(q)$ as:

$$D_2(q) = \left\{ b = (b_k) \in \omega : \sum_{j=k}^{\infty} s_q(j, k) B_q(k) b_j \text{ exists, } \forall k \in \mathbb{N} \right\},$$

$$D_3(q) = \left\{ b = (b_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=0}^n \left| \sum_{j=k}^n s_q(j, k) B_q(k) b_j \right|^r < \infty \right\},$$

$$D_4(q) = \left\{ b = (b_k) \in \omega : \lim_{n \rightarrow \infty} \sum_{k=0}^n \left| \sum_{j=k}^n s_q(j, k) B_q(k) b_j \right| = \sum_k \left| \sum_{j=k}^{\infty} s_q(j, k) B_q(k) b_j \right| < \infty \right\}.$$

Then, we have

- (a) $(\ell_p(\mathbb{S}_q))^\beta = D_2(q) \cap D_3(q)$ and
- (b) $(\ell_\infty(\mathbb{S}_q))^\beta = D_2(q) \cap D_4(q)$ for $1 < p < \infty$.
- (c) $(\ell_p(\mathbb{S}_q))^\gamma = D_3(q)$, for $1 < p \leq \infty$.

Proof. (a) Let us take $b = (b_k) \in \omega$ and look at the equality

$$\sum_{k=0}^n b_k x_k = \sum_{k=1}^n b_k \left(\sum_{j=0}^n s_q(j, k) B_q(k) y_j \right) = \sum_{k=0}^n \left(\sum_{j=k}^n s_q(j, k) B_q(k) b_j \right) y_k = D_n(y), \quad (15)$$

where $D = (d_{nk})$ is determined by

$$d_{nk} = \begin{cases} \sum_{j=k}^n s_q(j, k) B_q(k) b_j, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

After that, we deduce from Lemma 2 (ii) using (2) that $Dy \in c$ whenever $y = (y_k) \in \ell_p$ if and only if $b x = (b_k x_k) \in cs$ whenever $x \in \ell_p(\mathbb{S}_q)$. Therefore, $(b_k) \in (\ell_p(\mathbb{S}_q))^\beta$ if and only if $(b_k) \in D_2(q)$ and $(b_k) \in D_3(q)$ are defined by (11) and (12), respectively. Consequently, $(\ell_p(\mathbb{S}_q))^\beta = D_2(q) \cap D_3(q)$.

(b) An equivalent proof can be formulated when $p = \infty$ by utilizing Lemma 2 (iii) in place of Lemma 2 (ii) through analogous approaches.

(c) One may utilize (15) to produce the proof by using Lemma 3. $\square$

We recommend that readers consult the book by Başar and Dutta [30] and the article by Kara [31] for a better understanding of the studies presented in this section.

4 Matrix transformations associated with the space $\ell_p(\mathbb{S}_q)$

The matrix classes $(\ell_p(\mathbb{S}_q), U)$ are characterized in this section, where $1 < p \leq \infty$, and $U \in \{\ell_\infty, \ell_1, c, c_0\}$. We utilize

$$\tilde{b}_{nk} = \sum_{j=k}^{\infty} s_q(j, k) B_q(k) b_{nj}$$

in order to achieve brevity.

The following lemma forms the basis of our findings.

Lemma 4. ([32], Theorem 4.1) Let μ be an arbitrary subset of ω , U a triangular matrix, V its inverse, and λ a FK-space. Define $H^{(n)} = (h_{mk}^{(n)})$ and $H = (h_{nk})$ by

$$H^{(n)} = h_{mk}^{(n)} = \begin{cases} \sum_{j=k}^m b_{nj} v_{jk}, & 0 \leq k \leq m, \\ 0, & k > m, \end{cases} \quad H = (h_{nk}) = \sum_{j=k}^{\infty} b_{nj} v_{jk},$$

respectively. Thus, we obtain $H^{(n)} = (h_{mk}^{(n)}) \in (\lambda, c)$ and $H = (h_{nk}) \in (\lambda, \mu)$ if and only if $B = (b_{nk}) \in (\lambda_U, \mu)$.

The following conditions are now listed:

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^m \left| \sum_{j=k}^m s_q(j, k) B_q(k) b_{nj} \right|^r < \infty, \quad (16)$$

$$\lim_{m \rightarrow \infty} \sum_{j=k}^m s_q(j, k) B_q(k) b_{nj} = \tilde{b}_{nk}, \quad \forall n, k \in \mathbb{N}, \quad (17)$$

$$\lim_{m \rightarrow \infty} \sum_{k=0}^m \left| \sum_{j=k}^m s_q(j, k) B_q(k) b_{nj} \right| = \sum_k |\tilde{b}_{nk}| \quad \forall n \in \mathbb{N}, \quad (18)$$

$$\sup_{m \in \mathbb{N}} \sum_k |\tilde{b}_{nk}|^r < \infty, \quad (19)$$

$$\sup_{N \in F} \sum_k \left| \sum_{n \in \mathbb{N}} \tilde{b}_{nk} \right|^r < \infty, \quad (20)$$

$$\lim_{n \rightarrow \infty} \tilde{b}_{nk} = \tilde{\alpha}_k; \quad k \in \mathbb{N}, \quad (21)$$

$$\lim_{n \rightarrow \infty} \sum_k |\tilde{b}_{nk}| = \sum_k |\tilde{\alpha}_k|, \quad (22)$$

$$\lim_{n \rightarrow \infty} \sum_k \tilde{b}_{nk} = 0, \quad (23)$$

$$\sup_{n, k \in \mathbb{N}} |\tilde{b}_{nk}| < \infty, \quad (24)$$

$$\sup_{k, m \in \mathbb{N}} \left| \sum_{j=k}^m s_q(j, k) B_q(k) b_{nj} \right| < \infty, \quad (25)$$

$$\sup_{k \in \mathbb{N}} \sum_n |\tilde{b}_{nk}| < \infty, \quad (26)$$

$$\sup_{N, K \in F} \left| \sum_{n \in N} \sum_{k \in K} \tilde{b}_{nk} \right| < \infty. \quad (27)$$

Thus, utilizing Lemma 4 and the findings in [29], we may deduce the following results from the given conditions.

Proposition 1. For $p = 1$

- (i) $B = (b_{nk}) \in (\ell_1(\mathbb{S}_q), \ell_\infty) \Leftrightarrow (17), (24), \text{ and } (25) \text{ hold.}$
- (ii) $B = (b_{nk}) \in (\ell_1(\mathbb{S}_q), c) \Leftrightarrow (17), (21), (24), \text{ and } (25) \text{ hold.}$
- (iii) $B = (b_{nk}) \in (\ell_1(\mathbb{S}_q), c_0) \Leftrightarrow (17), \text{ with } \tilde{\alpha}_k = 0, (21), (24), \text{ and } (25) \text{ hold.}$
- (iv) $B = (b_{nk}) \in (\ell_1(\mathbb{S}_q), \ell_1) \Leftrightarrow (17), (25), \text{ and } (26) \text{ hold.}$

For $1 < p < \infty$,

- (v) $B = (b_{nk}) \in (\ell_p(\mathbb{S}_q), \ell_\infty) \Leftrightarrow (16), (17), \text{ and } (19) \text{ hold.}$
- (vi) $B = (b_{nk}) \in (\ell_p(\mathbb{S}_q), c) \Leftrightarrow (16), (17), (19), \text{ and } (21) \text{ hold.}$
- (vii) $B = (b_{nk}) \in (\ell_p(\mathbb{S}_q), c_0) \Leftrightarrow (16), (17), (19), \text{ and with } \tilde{\alpha}_k = 0 (21) \text{ hold.}$
- (viii) $B = (b_{nk}) \in (\ell_p(\mathbb{S}_q), \ell_1) \Leftrightarrow (16), (17), \text{ and } (20) \text{ hold.}$

For $p = \infty$

(ix) $B = (b_{nk}) \in (\ell_\infty(\mathbb{S}_q), \ell_\infty) \Leftrightarrow (17), (18)$ and in case $r = 1$ (19) hold.

(x) $B = (b_{nk}) \in (\ell_\infty(\mathbb{S}_q), c) \Leftrightarrow (17), (18), (21),$ and (22) hold.

(xi) $B = (b_{nk}) \in (\ell_\infty(\mathbb{S}_q), c_0) \Leftrightarrow (17), (18),$ and (23) hold.

(xii) $B = (b_{nk}) \in (\ell_\infty(\mathbb{S}_q), \ell_1) \Leftrightarrow (17), (18),$ and (27) hold.

5 The $\ell_p(\mathbb{S}_q)$'s certain geometric characteristics

The geometric properties of Banach spaces – such as uniform convexity, strict convexity, reflexivity, the Banach-Saks property, the Kadec-Klee property, and uniform smoothness-play a critical role in understanding the structure of these spaces. These geometric properties determine analytical behaviors in Banach spaces, including convergence, compactness, and the solvability of optimization problems. In particular, uniform convexity and reflexivity form the foundation of many theorems in functional analysis. Hudzik et al. [33] calculated Banach-Saks type for two types of Banach sequence spaces and examined the convexity structure using Gurarii's modulus of convexity. Knaust [34] proved the existence of p -Hilbertian subsequences within any weakly null sequence in Orlicz sequence spaces possessing Banach-Saks property of type p . However, there is a profound relationship between the geometric properties of Banach spaces and fixed point theory. Specifically, the existence of a fixed point for an operator and its convergence to that point are significantly influenced by the geometry of the space, relying on the convexity, completeness, and compactness properties of Banach spaces. In particular, uniformly convex or reflexive spaces facilitate the existence of fixed points for nonexpansive operators. García-Falset [35] established that the class of weakly nearly uniformly smooth Banach spaces has the fixed point property relative to nonexpansive mappings. Also, Mursaleen et al. [36] showed that the Euler sequence spaces are uniformly and strictly convex and have the weak fixed point property.

We will discuss $\ell_p(\mathbb{S}_q)$'s some geometric properties for $1 < p < \infty$ in this section.

If every bounded sequence (x_n) in D admits a subsequence (r_n) such that $\{t_k(r)\}$ is convergent in the norm in D , a Banach space D has the Banach-Saks property [36], where

$$\{t_k(r)\} = \frac{1}{k+1}(r_0 + r_1 + \dots + r_k) \quad (k \in \mathbb{N}). \quad (28)$$

A Banach space D has the weak Banach-Saks property for any given weakly null sequence $(s_n) \subset D$ if there exists a subsequence (r_n) of (s_n) such that $\{t_k(r)\}$ is strongly convergent to zero.

The coefficient that García-Falset provides in [35] is as follows:

$$R(D) = \sup \left\{ \liminf_{n \rightarrow \infty} \|s_n - s\| : (s_n) \subset B(D), s_n \xrightarrow{w} s, s \in B(D) \right\}, \quad (29)$$

where $B(D)$ denotes the unit ball of D .

Remark 1. For $R(D) < 2$, a Banach space D has the weak fixed point property [35].

For $\forall n \in \mathbb{N}$, $1 < p < \infty$ and some $M > 0$, if every weakly null sequence (s_k) has a subsequence (s_{k_l}) such that

$$\left\| \sum_{l=0}^n s_{k_l} \right\| < M(n+1)^{1/p}, \quad (30)$$

a Banach space has the Banach-Saks type p or the property $(BS)_p$ [34].

$\ell_p(\mathbb{S}_q)$'s some geometric properties, where $1 < p < \infty$, allow us to calculate the following conclusions.

Theorem 11. $\ell_p(\mathbb{S}_q)$ ($1 < p < \infty$) has the Banach-Saks type p .

Proof. Let us take (ε_n) sequence such that $(\varepsilon_n) > 0$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \varepsilon_n \leq \frac{1}{2}$, and additionally, we assume a weakly null sequence (s_n) in $B(\ell_p(\mathbb{S}_q))$. Set $r_0 = s_0 = 0$ and $r_1 = s_{n_1} = s_1$. After that, a $u_1 \in \mathbb{N}$ exists such that

$$\left\| \sum_{i=u_1+1}^{\infty} r_1(i)e^{(i)} \right\|_{\ell_p(\mathbb{S}_q)} < \varepsilon_1. \quad (31)$$

There is an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=0}^{u_1} s_{n_2}(i)e^{(i)} \right\|_{\ell_p(\mathbb{S}_q)} < \varepsilon_1 \quad (32)$$

when $n \geq n_2$, because (s_n) is a weakly null sequence that implies $s_n \rightarrow 0$ coordinate wise. Set $r_2 = s_{n_2}$. Then, there exists an $u_2 > u_1$ such that

$$\left\| \sum_{i=u_2+1}^{\infty} r_2(i)e^{(i)} \right\|_{\ell_p(\mathbb{S}_q)} < \varepsilon_2. \quad (33)$$

Applying once more the understanding that $s_n \rightarrow 0$ coordinate wise, there exists an such that $n_3 > n_2$

$$\left\| \sum_{i=0}^{u_2} s_{n_3}(i)e^{(i)} \right\|_{\ell_p(\mathbb{S}_q)} < \varepsilon_2, \quad (34)$$

when $n \geq n_3$.

Two increasing subsequences (u_i) and (n_i) may be identified if we carry out this procedure further, such that

$$\left\| \sum_{i=0}^{u_j} s_{n_j}(i)e^{(i)} \right\|_{\ell_p(\mathbb{S}_q)} < \varepsilon_j, \quad (35)$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=u_j+1}^{\infty} r_j(i)e^{(i)} \right\|_{\ell_p(\mathbb{S}_q)} < \varepsilon_j. \quad (36)$$

where $r_j = s_{n_j}$. Thus,

$$\begin{aligned} \left\| \sum_{j=0}^n r_j \right\|_{\ell_p(\mathbb{S}_q)} &= \left\| \sum_{j=0}^n \left(\sum_{i=0}^{u_{j-1}} r_j(i)e^{(i)} + \sum_{i=u_{j-1}+1}^{u_j} r_j(i)e^{(i)} + \sum_{i=u_j+1}^{\infty} r_j(i)e^{(i)} \right) \right\|_{\ell_p(\mathbb{S}_q)} \\ &\leq \left\| \sum_{j=0}^n \left(\sum_{i=u_{j-1}+1}^{u_j} r_j(i)e^{(i)} \right) \right\|_{\ell_p(\mathbb{S}_q)} + 2 \sum_{j=0}^n \varepsilon_j. \end{aligned}$$

The other side, $\|s\|_{\ell_p(\mathbb{S}_q)} \leq 1$ is apparent. Thus, we have that

$$\begin{aligned} &\left\| \sum_{j=0}^n \left(\sum_{i=u_{j-1}+1}^{u_j} r_j(i)e^{(i)} \right) \right\|_{\ell_p(\mathbb{S}_q)}^p \\ &= \sum_{j=0}^n \sum_{i=u_{j-1}+1}^{u_j} \left| \frac{1}{B_q(i)} \sum_{k=0}^i S_q(i, k) r_j(i) \right|^p \\ &\leq \sum_{j=0}^n \sum_{i=0}^{\infty} \left| \frac{1}{B_q(i)} \sum_{k=0}^i S_q(i, k) r_j(i) \right|^p \leq (n+1). \end{aligned}$$

Hence, it can be obtained that

$$\left\| \sum_{j=0}^n \left(\sum_{i=u_{j-1}+1}^{u_j} r_j(i) e^{(i)} \right) \right\|_{\ell_p(\mathbb{S}_q)} \leq (n+1)^{\frac{1}{p}}.$$

Utilizing the knowledge that $1 \leq (n+1)^{\frac{1}{p}}$ for $1 < p < \infty$ and $\forall n \in \mathbb{N}$, we obtain

$$\left\| \sum_{j=0}^n r_j \right\|_{\ell_p(\mathbb{S}_q)} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}.$$

As a consequence, $\ell_p(\mathbb{S}_q)$ has the Banach-Saks type p . The proof is completed now. $\square$

Remark 2. $R(\ell_p(\mathbb{S}_q)) = R(\ell_p) = 2^{\frac{1}{p}}$, because $\ell_p(\mathbb{S}_q)$ is linearly isomorphic to ℓ_p .

We derive the following result from Remarks 1 and 2.

Corollary 1. *The space $\ell_p(\mathbb{S}_q)$ ($1 < p < \infty$) satisfies the weak fixed-point property.*

6 Conclusion

Cesaro, Zweier, Abel, Borel, and related matrices occupy a significant place in summability theory. In recent years, q -Cesaro, q -Fibonacci, q -Pascal, and q -Catalan matrices have been developed using q -analogs, and various properties of the corresponding sequence spaces have been investigated. In this study, we introduced the spaces $c_0(\mathbb{S}_q)$, $c(\mathbb{S}_q)$, $\ell_\infty(\mathbb{S}_q)$, and $\ell_p(\mathbb{S}_q)$ ($1 \leq p < \infty$) by means of matrices constructed via q -Stirling and q -Bell numbers. We presented some topological and geometric properties of these spaces and characterized certain matrix classes. Among the current topics of interest in summability theory are the concepts of statistical convergence and statistical boundedness. We believe that in the near future, researchers will further explore these notions for sequences defined using q -Stirling and q -Bell numbers.

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