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Research Article

A New Method for Solving Sequential Fractional Wave Equations

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In this article, we focus on two classes of fractional wave equations in the context of the sequential Caputo derivative. For the first class, we derive the closed-form solution in terms of generalized Mittag–Leffler functions. Subsequently, we consider a more general class of nonhomogeneous fractional wave equations. Due to the complexity of finding exact solutions for these problems, we employ a numerical technique based on the operational matrix method to approximate the solution. We provide several theoretical and numerical examples to validate the effectiveness of this numerical approach. The results demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

Fractional partial differential equations (FPDEs) have received increasing attention over the last few decades due to their applications in various fields, including physics, chemistry, engineering, and economics. Sequential fractional derivatives are a type of fractional derivative that involves the application of the fractional differentiation operator multiple times in sequence. In other words, the output of the first fractional differentiation operator becomes the input to the next fractional differentiation operator. The sequential fractional derivative provides a more accurate description of complex systems, such as biological systems, exhibiting nonlocal and non-Markovian behavior. It also captures long-term memory effects that are not captured by integer-order derivatives. This makes it useful for modeling systems with long-term dependencies. Moreover, it is more flexible than other kinds of derivatives because it allows for a continuous range of fractional values, enabling a more precise modeling of systems with complex behavior. Sequential fractional derivatives have been successfully used in a wide range of applications, including control theory, signal processing, and image processing. They have been studied extensively in recent years due to their potential applications in various areas of science and engineering. For example, they have been used to model anomalous diffusion phenomena in complex systems such as porous media, fractal structures, and biological tissues. They have also been used in the analysis of viscoelastic materials and in the design of control systems.

Several numerical methods have been proposed for the approximation of sequential fractional derivatives. One common approach is to use the recursive formula to compute the derivatives in sequence. Another approach is to use the fractional integration operator to convert the sequential fractional derivative into a single fractional derivative, which can then be approximated using existing numerical methods for fractional derivatives. Overall, sequential fractional derivatives are a powerful tool for modeling and analyzing complex systems, and their study is an active area of research in the field of fractional calculus.

In [1], Yang introduces the concept of sequential fractional derivatives and discusses their properties and applications. The author provides some examples to illustrate the use of sequential fractional differential equations in modeling complex systems. In [2], the authors study the solutions of sequential fractional differential equations and provide numerical examples to illustrate their results. They also introduce a new approach for solving sequential fractional differential equations using the fractional Laplace transform. Torres [3] introduces the concept of sequential fractional variational calculus, which is an extension of the standard fractional variational calculus to include sequential fractional derivatives. The author discusses the properties of sequential fractional variational calculus and provides some examples to illustrate its applications.

The operational matrix method is a popular numerical technique for solving FPDEs. In [4], the authors developed a new approach for solving two-dimensional FPDEs using a combination of the operational matrix method and the Galerkin method. The proposed approach was applied to a range of problems, including the fractional diffusion equation. The numerical results demonstrated the efficiency and accuracy of the method.

Another study by Yousefi and Delavari [5] focused on the application of the operational matrix method to solve a class of nonlinear FPDEs. The authors developed a new technique that combines the operational matrix method with the Newton–Raphson method to solve a variety of nonlinear FPDEs. In [6], the authors developed a new approach for solving time-fractional diffusion equations using a combination of the operational matrix method and the Tau method. The proposed approach was applied to several problems, including the time-fractional diffusion equation with a space-dependent coefficient. Moreover, Hesameddini et al. [7] proposed a new numerical technique based on the operational matrix method to solve multiterm time-fractional partial differential equations. The authors applied the proposed method to a variety of problems, including the multiterm time-fractional diffusion equation and the multiterm time-fractional wave equation.

Finally, in [8], a numerical technique based on the operational matrix method was developed to solve the time-fractional diffusion equation in two dimensions. The authors used the proposed method to solve several problems, including the time-fractional diffusion equation with a space-dependent coefficient. In [9], the authors developed a new numerical technique based on the operational matrix method to solve the time-fractional advection-diffusion equation. The proposed method was shown to be accurate and efficient for solving this type of equation. In [10], the authors proposed a new approach based on the operational matrix method to solve the space-time fractional diffusion equation. The proposed method was compared with several other numerical methods, and the results showed that it is accurate and efficient. In [11], the authors developed a new numerical technique based on the operational matrix method to solve the multiterm time-fractional diffusion equation. The proposed method was shown to be accurate and efficient, outperforming several other numerical methods. In [12], the authors developed a new numerical technique based on the operational matrix method to solve the fractional telegraph equation. The proposed method was shown to be accurate and efficient and was compared with several other numerical methods. In [13], the authors developed a new approach based on the operational matrix method to solve the fractional diffusion equation with a space-dependent coefficient. Several researchers have investigated the applications of fractional derivatives and developed numerical methods for solving such applications. In [14], extended fractional dynamical systems are studied, while [15] focuses on multispace fractional Korteweg–de Vries Equations. The chaotic Lorenz system is solved using wavelet polynomials in [16], and [17] addresses the solution of delay differential equations. In [18, 19], the collocation method and series method are employed to solve fractional initial value problems (IVPs), respectively. However, [20] presents an iterative method developed for solving such problems. The operational matrix method is used to solve systems of fractional IVPs in [21]. The existence and uniqueness of fractional problems are examined in [22].

In this article, we compare and contrast the sequential Caputo derivative and the Caputo derivative. We extend these concepts to solve a class of homogeneous fractional wave equations with constant coefficients. We derive the analytical solution for a specific case of a homogeneous second-order fractional equation with constant coefficients. We provide evidence to support our findings and present two practical illustrations of their application.

Additionally, we address a different category of nonhomogeneous fractional wave equations with variable coefficients. Since finding exact solutions for these problems is challenging and sometimes not feasible, we employ a numerical technique for approximate solutions. The operational matrix method serves as the foundation for this approach. We derive the operational matrices to integrate the equations into a unified system. This method offers advantages such as low equation setup costs and avoids the need for projection methods like Galerkin or collocation.

To validate the effectiveness of this approach, we provide two examples for experimental testing. The results demonstrate a close match between the exact and approximation solutions, as observed in the graphs. Furthermore, the L_2-errors for various values of t are nearly zero, indicating the accuracy of the method.

For solving fractional partial differential equations, the operational matrix method has several advantages. The operational matrix method is an easy-to-understand technique for solving differential equations. It transforms the task of solving a differential equation into solving a set of algebraic equations, which can be efficiently and accurately solved using numerical methods. It is highly efficient as it avoids directly solving the differential equation and simplifies the problem into solving a system of algebraic equations.

Moreover, the operational matrix method provides precise solutions for a wide range of problems. It is flexible and can be applied to various types of differential equations, making it suitable for diverse applications. Additionally, the operational matrix method can be combined with other numerical methods to enhance its capabilities and address specific requirements.

Furthermore, the operational matrix method has a strong theoretical foundation, with well-established principles and extensive research support. This solid theoretical basis makes the operational matrix method a popular choice in scientific and engineering applications. Its simplicity, efficiency, and accuracy contribute to its widespread use in various fields. All of the aforementioned advantages motivate us to study and investigate the solution to this problem.

This research paper is divided into six sections. In sections one and two, we provide a brief literature review of fractional wave equations and the operational matrix method. Then, we present some definitions and results necessary for this paper. In section three, we provide a detailed description of the exact solution for special types of fractional problems. We find the fundamental set of solutions in the closed form for a class of fractional wave equations. In section four, we derive the numerical technique used to solve a class of nonhomogeneous wave equations. Proofs of our results are given in sections three and four. Several examples are provided in section five to demonstrate the efficiency of the proposed method. Finally, we summarize the main findings of this study and draw conclusions in section six.

2. Basic Definitions and Results

Now, let's begin with the definition of the Caputo derivative and the fractional integral operator.

Definition 1 (see [23]). Let $L \in \mathbb{N}$ and $p > 0$. Define the spaces C_p and C_p^L as

$$\begin{aligned} C_p &= \left\{w:\ (0,\infty) \longrightarrow \mathbb{R}:\ w(x) = x^j w_1(x), w_1 \in C[0,\infty), j > p\right\},\\ C_p^L &= \left\{w:\ (0,\infty) \longrightarrow \mathbb{R}:\ w^{(L)} \in C_p\right\}. \end{aligned} \tag{1}$$

If $L-1 < p < L,\ t > 0,\ w \in C_{-1}^L$, then the Caputo derivative is given by

$${}^cD^p w(z) = \frac{1}{\Gamma(L-p)} \int_0^z (z-r)^{n-p-1} w^{(n)}(r) \mathrm{d}r, \tag{2}$$

where Γ is the Gamma function, and the fractional integral operator is defined by

$$I^p w(z) = \frac{1}{\Gamma(p)} \int_0^z (z-r)^{p-1} w(r) \mathrm{d}r, \tag{3}$$

where $L-1 < p < L$ and $L \in \mathbb{N}$.

The first important rule is the power rule, which is given by

$${}^cD^p z^q = \left\{ \begin{array}{cc} 0, & q < p, q \in \{0,1,2,\cdots\} \\ \dfrac{\Gamma(q+1)}{\Gamma(q-p+1)} z^{q-p}, & \text{otherwise} \end{array} \right\}. \tag{4}$$

The composites of the Caputo derivative and fractional integral operator are given by the following relations:

$$\begin{aligned} {}^cD^p I^p w(z) &= w(z),\\ I^p\left({}^cD^p w(z)\right) &= w(z) - w(0). \end{aligned} \tag{5}$$

If $(1/2) < q \leq 1$ and $L \in \mathbb{N}$, the power rule implies that

$$\begin{aligned} {}^cD^{2p} z^p &= \frac{\Gamma(p+1)}{\Gamma(2p-p+1)} z^{p-2p}\\ &= \frac{\Gamma(p+1)}{\Gamma(1-p)} z^{-p}. \end{aligned} \tag{6}$$

However,

$$\begin{aligned} {}^cD^p\left({}^cD^p z^p\right) &= {}^cD^p\left(\frac{\Gamma(p+1)}{\Gamma(1)}\right)\\ &= 0. \end{aligned} \tag{7}$$

From equations (6) and (7), one can see that

$${}^cD^{2p} w(z) \neq {}^cD^p\left({}^cD^p w(z)\right), \tag{8}$$

in general. In this case, we say that the Caputo derivative is not sequential. The formal definition of the sequential Caputo derivative is given as follows.

Definition 2 (see [24]). If $L-1 < Lp < L$ and

$${}^cD^{Lp} w(z) = {}^cD^p\left({}^cD^{(L-1)p} w(z)\right), \tag{9}$$

for $L \in \{2,3,\ldots\}$, *then the Caputo derivative is called the sequential Caputo derivative of order p. In this case, we use the following notation* ${}^{sc}D^{Lp} w(z)$.

This property is an important property, and it will affect the solution of the fractional wave equations. Before we go further in this paper, we need the following result.

Theorem 3. *Let* $E_p(z) = \sum_{j=0}^{\infty} (z^j/\Gamma(jp+1))$ *be the Mittag–Leffler function. If* $0 < p < 1$ *and* μ *be any constant, then*

$${}^cD^{2p} E_p(\mu z^p) \neq \mu^2 E_p(\mu z^p), \tag{10}$$

while

$${}^{sc}D^{2p} E_p(\mu z^p) = \mu^2 E_p(\mu z^p). \tag{11}$$

Proof. Formula (10) follows directly from equation (6). Now,

$$\begin{aligned}{}^{sc}D^{2p}E_p(\mu z^p) \neq {}^{sc}D^{2p}\left(\sum_{j=0}^{\infty}\frac{\mu^j z^{pj}}{\Gamma(jp+1)}\right) &= \sum_{j=2}^{\infty}\frac{\mu^j z^{p(j-2)}}{\Gamma((j-2)p+1)} \\ &= \mu^2\sum_{j=0}^{\infty}\frac{\mu^j z^{pj}}{\Gamma(jp+1)} \\ &= \mu^2 E_p(\mu z^p).\end{aligned} \tag{12}$$

□

For more details about fractional derivatives and their properties, we refer the reader to the following references. In [23], the properties of fractional derivatives and some of their applications are discussed. Additionally, Podlubny [25] provides a theoretical discussion on fractional calculus. In [24], the authors compare sequential and nonsequential fractional derivatives. From now on, we use the following notation D^p to mean ${}^{sc}D^p$.

We study the exact and numerical solutions of the following class of fractional wave equations in this article

$$\begin{aligned}&D_t^{2p}v(x,t)+\alpha(t)D_t^p v(x,t)+\beta(t)v(x,t)\\ &\quad = \kappa(x)v_{xx}(x,t)+h(x,t),\end{aligned} \tag{13}$$

with

$$\begin{aligned}v(0,t) &= v(\eta,t)\\ &= 0, t\geq 0,\end{aligned} \tag{14}$$

$$\begin{aligned}v(x,0) &= y(x),\\ D_t^p v(x,0) &= 0, 0\leq x\leq\eta,\end{aligned} \tag{15}$$

where $\alpha(t)$ and $\beta(t)$ are continuous functions in $[0,\infty)$, $(1/2)<p\leq 1$, $y\in C^3[0,\eta]$, $h\in C([0,\eta]\times[0,\infty))$, and $\kappa\in C[0,\eta]$.

3. Homogeneous Fractional Wave Equation with Constant Coefficients

In this section, we investigate the exact solution of the following special case of problems (13)-(14)

$$D_t^{2p}v(x,t)+\alpha D_t^p v(x,t)+\beta v(x,t) = v_{xx}(x,t), \tag{16}$$

with

$$\begin{aligned}v(0,t) &= v(\eta,t)\\ &= 0, t\geq 0,\end{aligned} \tag{17}$$

where α and β are real constants. Now, we start with the first result which is given by the following theorem.

Theorem 4. *Let* $(1/2)<p\leq 1$. *If* $E_{p,p}(z)=\sum_{j=0}^{\infty}(z^j/\Gamma(jp+p))$ *be the generalized Mittag–Leffler function, then*

$$u(t) = t^p E_{p,p}(\mu t^p), \tag{18}$$

is a solution to the following fractional problem

$$D^{2p}u(t)+\alpha D^p u(t)+\beta u(t) = 0, \tag{19}$$

when $\alpha^2 = 4\beta$ *and* $\mu = (-\alpha/2)$.

Proof. Using the fact $\Gamma(z+1)=z\Gamma(z)$ and equation (4), we get

$$\begin{aligned}D^p\left(t^pE_{p,p}(\mu t^p)\right) &= D^p\left(\sum_{j=0}^{\infty}\frac{\mu^j t^{(j+1)p}}{\Gamma(jp+p)}\right)\\ &= \sum_{j=0}^{\infty}\frac{\mu^j\Gamma(jp+p+1)t^{jp}}{\Gamma(jp+1)\Gamma(jp+p)}\\ &= \sum_{j=0}^{\infty}\frac{\mu^j p(j+1)t^{jp}}{\Gamma(jp+1)},\\ D^{2p}\left(t^pE_{p,p}(\mu t^p)\right) &= D^p\left(\sum_{j=0}^{\infty}\frac{\mu^j p(j+1)t^{jp}}{\Gamma(jp+1)}\right)\\ &= \sum_{j=1}^{\infty}\frac{\mu^j p(j+1)t^{(j-1)p}}{\Gamma((j-1)p+1)}\\ &= \sum_{j=0}^{\infty}\frac{\mu^{j+1}p(j+2)t^{jp}}{\Gamma(jp+1)}.\end{aligned} \tag{20}$$

Then,

$$D^{2p}u(t)+\alpha D^p u(t)+\beta u(t) = \sum_{j=0}^{\infty}\frac{\mu^{j+1}p(j+2)t^{jp}}{\Gamma(jp+1)}+\alpha\sum_{j=0}^{\infty}\frac{\mu^j p(j+1)t^{jp}}{\Gamma(jp+1)}+\beta\sum_{j=0}^{\infty}\frac{\mu^j t^{(j+1)p}}{\Gamma(jp+p)}, \tag{21}$$

which can be written as

$$D^{2p}u(t)+\alpha D^p u(t)+\beta u(t) = \sum_{j=0}^{\infty}\frac{\mu^{j+1}p(j+2)t^{jp}}{\Gamma(jp+1)}+\alpha\sum_{j=0}^{\infty}\frac{\mu^j p(j+1)t^{jp}}{\Gamma(jp+1)}+\beta\sum_{j=1}^{\infty}\frac{\mu^{j-1}t^{jp}}{\Gamma(jp)}. \tag{22}$$

For $j = 1, 2, \ldots$, the coefficient of t^{jp} is simplified as

$$\begin{aligned}
\frac{\mu^{j+1} p(j+2)}{\Gamma(jp+1)} + \alpha\frac{\mu^{j} p(j+1)}{\Gamma(jp+1)} + \beta\frac{\mu^{j-1}}{\Gamma(jp)} &= \frac{\mu^{j+1}(j+2)}{j\Gamma(jp)} + \alpha\frac{\mu^{j}(j+1)}{j\Gamma(jp)} + \beta\frac{\mu^{j-1}}{\Gamma(jp)} \\
&= \mu^{j-1}\frac{\mu^2(j+2) + \alpha\mu(j+1) + \beta j}{j\Gamma(jp)} \\
&= j\mu^{j-1}\frac{\mu^2 + \alpha\mu + \beta}{\Gamma(jp)} + \mu^{j-1}\frac{2\mu^2 + \alpha\mu}{j\Gamma(jp)} \\
&= 0.
\end{aligned} \tag{23}$$

Since $\mu^2 + \alpha\mu + \beta = 0$ and $\mu = (-\alpha/2)$. For $j = 0$, we have

$$\begin{aligned}
2p\mu + p\alpha &= p(2\mu + \alpha) \\
&= 0,
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
&D^{2p}\left(t^p E_{p,p}(\mu t^p)\right) + \alpha D^p\left(t^p E_{p,p}(\mu t^p)\right) \\
&\quad + \beta t^p E_{p,p}(\mu t^p) = 0,
\end{aligned} \tag{25}$$

which completes the proof. □

Theorem 5. *Let $\alpha \neq 0$ and β be two constants and $(1/2) < p \leq 1$. Then, the exact solution of the following fractional differential equation:*

$$D^{2p}u(t) + \alpha D^p u(t) + \beta u(t) = 0, \tag{26}$$

is given by

$$u(t) = \begin{cases} c_1 E_p(\mu_1 t^p) + c_2 E_p(\mu_2 t^p), \alpha^2 \neq 4\beta, \\ c_1 E_p\left(\dfrac{-\alpha}{2}t^p\right) + c_2 t^p E_{p,p}\left(\dfrac{-\alpha}{2}t^p\right), \alpha^2 = 4\beta, \end{cases} \tag{27}$$

where

$$\begin{aligned}
\mu_1 &= \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \\
\mu_2 &= \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}.
\end{aligned} \tag{28}$$

Proof. We divided the proof into two cases. In the first case, we assume that $\alpha^2 \neq 4\beta$. Then, simple calculations imply that

$$\begin{aligned}
\left(\mu_1^2 + \alpha\mu_1 + \beta\right) &= 0, \\
\left(\mu_2^2 + \alpha\mu_2 + \beta\right) &= 0,
\end{aligned} \tag{29}$$

which yield using equation (11) to the following equations:

$$\begin{aligned}
D^{2p}E_p(\mu_1 t^p) + \alpha D^p E_p(\mu_1 t^p) + \beta E_p(\mu_1 t^p) &= \left(\mu_1^2 + \alpha\mu_1 + \beta\right)E_p(\mu_1 t^p) \\
&= 0, \\
D^{2p}E_p(\mu_2 t^p) + \alpha D^p E_p(\mu_2 t^p) + \beta E_p(\mu_2 t^p) &= \left(\mu_2^2 + \alpha\mu_2 + \beta\right)E_p(\mu_2 t^p) \\
&= 0.
\end{aligned} \tag{30}$$

Thus, $E_p(\mu_1 t^p)$ and $E_p(\mu_2 t^p)$ are solutions to equation (26). Now, we want to show that they are linearly independent. To show that, we set $c_1 E_p(\mu_1 t^p) + c_2 E_p(\mu_2 t^p) = 0$. Then,

$$\begin{aligned}
c_1 E_p(\mu_1 t^p) + c_2 E_p(\mu_2 t^p) &= \sum_{j=0}^{\infty}\left(c_1\mu_1^j + c_2\mu_2^j\right)t^{jp} \\
&= 0,
\end{aligned} \tag{31}$$

which implies that

$$c_1\mu_1^j + c_2\mu_2^j = 0, j = 0, 1, 2, \cdots. \tag{32}$$

For $j = 0$ and 1, we get

$$\begin{pmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{33}$$

Since

$$\det\begin{pmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{pmatrix} = \mu_2 - \mu_1 \neq 0, \tag{34}$$

then $c_1 = c_2 = 0$. Thus, $E_p(\mu_1 t^p)$ and $E_p(\mu_2 t^p)$ are linearly independent. Now, we consider the case when $\alpha^2 = 4\beta$. Then, using Theorem 4, we have

$$\begin{aligned} &D^{2p}E_p\left(\frac{-\alpha}{2}t^p\right) + \alpha D^p E_p\left(\frac{-\alpha}{2}t^p\right) + \beta E_p\left(\frac{-\alpha}{2}t^p\right) = \left(\frac{\alpha^2}{4} - \frac{\alpha^2}{2} + \beta\right)E_p\left(\frac{-\alpha}{2}t^p\right) = 0, \\ &D^{2p}t^p E_{p,p}\left(\frac{-\alpha}{2}t^p\right) + \alpha D^p t^p E_{p,p}\left(\frac{-\alpha}{2}t^p\right) + \beta t^p E_{p,p}\left(\frac{-\alpha}{2}t^p\right) = 0. \end{aligned} \tag{35}$$

Then, $E_p((-\alpha/2)t^p)$ and $t^p E_{p,p}((-\alpha/2)t^p)$ are solutions to the problem (26). Now, let

$$c_1 E_p\left(\frac{-\alpha}{2}t^p\right) + c_2 t^p E_{p,p}\left(\frac{-\alpha}{2}t^p\right) = 0. \tag{36}$$

When $t = 0$, direct substitution in equation (36) implies that $c_1 = 0$. When $t = 1$, we will get $c_2 E_{p,p}(-\alpha/2) = 0$ which implies that $c_2 = 0$. This completes the proof of the theorem. □

Now, we discuss the solution of problems (16)-(17) in the following theorem.

Theorem 6. *Let $\alpha \neq 0$ and β be two constants, $(1/2) < p \leq 1$. Then, the exact solution of the following fractional problem*

$$D_t^{2p}v(x,t) + \alpha D_t^p v(x,t) + \beta v(x,t) = v_{xx}(x,t), \tag{37}$$

with

$$\begin{aligned} v(0,t) &= v(\eta,t) \\ &= 0, t \geq 0, \end{aligned} \tag{38}$$

is given as

$$v(x,t) = \sum_{j=1}^{\infty} u_j(t)\sin\left(\frac{j\pi}{\eta}x\right), \tag{39}$$

where

$$\begin{aligned} u_j(t) &= \begin{cases} a_j E_p(\mu_1 t^p) + b_j E_p(\mu_2 t^p), \alpha^2 \neq 4(\beta + \lambda_j), \\ a_j E_p\left(\frac{-\alpha}{2}t^p\right) + b_j t^p E_{p,p}\left(\frac{-\alpha}{2}t^p\right), \alpha^2 = 4(\beta + \lambda_j), \end{cases} \\ \mu_1 &= \frac{-\alpha + \sqrt{\alpha^2 - 4\beta - 4\lambda_j}}{2}, \\ \mu_2 &= \frac{-\alpha - \sqrt{\alpha^2 - 4\beta - 4\lambda_j}}{2}, \\ \lambda_j &= -\frac{j^2\pi^2}{\eta^2}, j \in \mathbb{N}. \end{aligned} \tag{40}$$

Proof. We use the separation of variables technique. Let

$$v(x,t) = u(t)v(x). \tag{41}$$

Then, substitute equation (39) into equation (37) to get

$$\begin{aligned} v''(x) &= \lambda v(x), \\ v(0) &= v(\eta) \\ &= 0, \end{aligned} \tag{42}$$

$$D_t^{2p}u(t) + \alpha D_t^p u(t) + (\beta - \lambda)u(t) = 0. \tag{43}$$

Equation (42) is an eigenvalue problem, and its eigenvalues and eigenfunctions are given as

$$v_j(x) = \sin\left(\frac{j\pi}{\eta}x\right),$$
$$\lambda_j = -\frac{j^2\pi^2}{\eta^2},\ j \in \mathbb{N}. \tag{44}$$

Theorem 5 implies that the solution of equation (43) is given as

$$u_j(t) = \begin{cases} a_j E_p(\mu_1 t^p) + b_j E_p(\mu_2 t^p), \alpha^2 \neq 4(\beta + \lambda_j), \\ a_j E_p\left(\frac{-\alpha}{2}t^p\right) + b_j t^p E_{p,p}\left(\frac{-\alpha}{2}t^p\right), \alpha^2 = 4(\beta + \lambda_j), \end{cases} \tag{45}$$

where

$$\mu_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta - 4\lambda_j}}{2},$$
$$\mu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta - 4\lambda_j}}{2},\ j \in \mathbb{N}. \tag{46}$$

Therefore, the solution of problems (37)-(38) is

$$v(x,t) = \sum_{j=1}^{\infty} u_j(t)v_j(x). \tag{47}$$

If we have initial conditions, we can use the Fourier series expansion to find a_j and b_j for $j \in \mathbb{N}$. □

4. Nonhomogeneous Fractional Wave Equation

In this section, we introduce a numerical technique for solving problems (13)–(15) with $\alpha(t) = 1$ and $\kappa(x) = 1$. The proposed numerical approach is based on the operational matrix method (OMM). Since the functions involved are two-dimensional, we define the two-dimensional block pulse functions (BPFs).

Definition 7. Let $\xi_1 = (\eta/L_1)$ and $\xi_2 = (T_{\max}/L_2)$ be two step-sizes where $L_1, L_2 \in \mathbb{N}$. Let $x_i = i\xi_1$ and $t_j = j\xi_2$ for $i \in I_{L_1} = \{0, 1, \cdots, L_1 - 1\}$ and $j \in I_{L_2} = \{0, 1, \cdots, L_2 - 1\}$. For any $i \in I_{L_1}$ and $j \in I_{L_2}$, the two-dimensional BPF is given by $\Xi_{i,j}$: $[0, \eta) \times [0, T_{\max}) \longrightarrow \mathfrak{R}$ where

$$\Xi_{i,j}(x,t) = \begin{cases} 1, & x \in [x_i, x_{i+1}), t \in [t_j, t_{j+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

Since $[x_{i_1}, x_{i_1+1}) \times [t_{j_1}, t_{j_1+1})$ and $[x_{i_2}, x_{i_2+1}) \times [t_{j_2}, t_{j_2+1})$ are disjoint sets, then simple calculations imply the following two properties of BPFs

$$\Xi_{i_1,j_1}(x,t)\Xi_{i_2,j_2}(x,t) = \begin{cases} \Xi_{i_1,j_1}(x,t), & i_1 = i_2, j_1 = j_2, \\ 0, & \text{otherwise,} \end{cases} \tag{49}$$

$$\int_0^{\eta}\int_0^{T_{\max}} \Xi_{i_1,j_1}(x,t)\Xi_{i_2,j_2}(x,t)\,dt\,dx = \begin{cases} \xi_1\xi_2, & i_1 = i_2, j_1 = j_2, \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

These two properties imply the result of this theorem.

Theorem 8. *Let v: $[0, \eta) \times [0, T_{\max}) \longrightarrow \mathfrak{R}$ be a square-integrable bounded function. Then,*

$$v(x,t) \approx \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} v_{i,j}\Xi_{i,j}(x,t), \tag{51}$$

where

$$v_{i,j} = \frac{1}{\xi_1\xi_2}\int_{x_i}^{x_i+1}\int_{t_j}^{t_j+1} v(x,t)\,dt\,dx. \tag{52}$$

Proof. Multiply both sides of equation (51) by $\Xi_{i,j}(x,t)$ and integrate on the domain $[0, \eta) \times [0, T_{\max})$. Then, the result of the theorem will follow directly from equations (49) and (50). □

One can see that $\Xi_{i,j}(x,t)$ can be written as a product of two BPFs as follows:

$$\Xi_{i,j}(x,t) = \varphi_i(x)\psi_j(t), \tag{53}$$

where $\{\varphi_i(x), i = 0, 1, \cdots, L_1 - 1\}$ and $\{\psi_j(t), j = 0, 1, \ldots, L_2 - 1\}$ are BPFs on $[0, \eta)$ and $[0, T_{\max})$, respectively. In this case, we can approximate the functions $v(x,t), \beta(t), h(x,t)$, and $y(x)$ as

$$v(x,t) \approx \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} v_{i,j}\varphi_i(x)\psi_j(t), \tag{54}$$

$$h(x,t) \approx \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} h_{i,j}\varphi_i(x)\psi_j(t), \tag{55}$$

$$\beta(t) \approx \sum_{j=0}^{L_2-1} \beta_j \psi_j(t),$$
$$y(x) \approx \sum_{i=0}^{L_1-1} y_i \varphi_i(x). \tag{56}$$

Then, using equation (54), we get

$$D_t^{2p} \nu(x,t) = \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x) D_t^{2p}\psi_j(t), \tag{57}$$

$$D_t^{p} \nu(x,t) = \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x) D_t^{p}\psi_j(t). \tag{58}$$

Also, equation (49) implies that

$$\beta(t)\nu(x,t) = \left(\sum_{j=0}^{L_2-1} \beta_j \psi_j(t)\right)\left(\sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x)\psi_j(t)\right) = \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \beta_j \nu_{i,j}\varphi_i(x)\psi_j(t). \tag{59}$$

From equations (54)-(55) and (57)–(59), we have

$$\sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x) D_t^{2p}\psi_j(t) + \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x) D_t^{p}\psi_j(t),$$
$$\sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \beta_j\nu_{i,j}\varphi_i(x)\psi_j(t) = \nu_{xx}(x,t) + \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} h_{i,j}\varphi_i(x)\psi_j(t). \tag{60}$$

To rewrite equation (60) in the matrix form, we define the following matrices:

$$\Phi = \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \vdots \\ \varphi_{L_1-1}(x) \end{pmatrix},$$

$$V = \begin{pmatrix} \nu_{00} & \nu_{01} & \cdots & \nu_{0(L_2-1)} \\ \nu_{10} & \nu_{11} & \cdots & \nu_{1(L_2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{(L_1-1)0} & \nu_{(L_1-1)1} & \cdots & \nu_{(L_1-1)(L_2-1)} \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \psi_0(t) \\ \psi_1(t) \\ \vdots \\ \psi_{L_2-1}(t) \end{pmatrix},$$

$$B = \begin{pmatrix} \beta_0 & 0 & \cdots & 0 \\ 0 & \beta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{L_2-1} \end{pmatrix},$$

$$Y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{L_1-1} \end{pmatrix},$$

$$H = \begin{pmatrix} h_{00} & h_{01} & \cdots & h_{0(L_2-1)} \\ h_{10} & h_{11} & \cdots & h_{1(L_2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{(L_1-1)0} & h_{(L_1-1)1} & \cdots & h_{(L_1-1)(L_2-1)} \end{pmatrix}. \tag{61}$$

Then, equation (60) becomes

$$\Phi^*(x)VD_t^{2p}\Psi(t) + \Phi^*(x)VD_t^{p}\Psi(t) + \Phi^*(x)VB\Psi(t) = \nu_{xx}(x,t) + \Phi^*(x)H\Psi(t), \tag{62}$$

where $*$ mean transpose of the matrix. From equations (54) and (58), we get

$$D_t^{p}\nu(x,0) = \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x) D_t^{p}\psi_j(0) = \Phi^*(x)VD_t^{p}\Psi(0) = 0,$$
$$\nu(x,0) = \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} \nu_{i,j}\varphi_i(x)\psi_j(0) = \Phi^*(x)V\Psi(0) = \Phi^*(x)Y, \tag{63}$$

which imply that

$$VD_t^{p}\Psi(0) = 0, \tag{64}$$

and

$$V\Psi(0) = Y. \tag{65}$$

Now, we need the operational integration matrix for the Riemann integral.

Theorem 9. *The operational matrix of the Riemann integral $I_x\Phi(x)$ is*

$$R_{I=(\xi_1/2)}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & \cdots & 2 & 2 \\ 0 & 0 & 1 & \ddots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{66}$$

Proof. For any $s \in \{0, 1, \ldots, L_1 - 1\}$, simple calculations imply that

$$\begin{aligned} I_x\varphi_s(x) &= \int_0^x \varphi_s(v)\mathrm{d}v \\ &= \begin{cases} 0, & x < s\xi_1, \\ x - s\xi_1, & s\xi_1 \le x < (s+1)\xi_1, \\ \xi_1, & (s+1)\xi_1 \le x < \eta. \end{cases} \end{aligned} \tag{67}$$

Therefore,

$$I_x\varphi_s(x) = \sum_{j=0}^{L_1-1} z_{j,s}\varphi_j(x), \tag{68}$$

implies that

$$\begin{aligned} z_{j,s} &= \frac{1}{\xi_1}\int_0^\eta (I_x\varphi_s(v))\varphi_j(v)\mathrm{d}v \\ &= \frac{1}{\xi_2}\int_{j\xi_2}^{(j+1)\xi_2} (I_x\varphi_s(v))\mathrm{d}v \\ &= \begin{cases} \dfrac{\xi_1}{2}, & 0 \le j = s \le L_1 - 1, \\ \xi_1, & 0 \le j < s \le L_1 - 1, \\ 0, & 0 \le s < j \le L_1 - 1, \end{cases} \end{aligned} \tag{69}$$

which gives the result of the theorem. □

Note that $I_x\Phi^*(x) = \Phi^*(x)R_I^*$. Using equations (62) and (66) and by taking the Riemann integral, we get

$$\Phi^*(x)R_I^*VD_t^{2p}\Psi(t) + \Phi^*(x)R_I^*VD_t^{p}\Psi(t) + \Phi^*(x)R_I^*VB\Psi(t) = v_x(x,t) - v_x(0,t) + \Phi^*(x)R_I^*H\Psi(t). \tag{70}$$

Let $v_x(0,t) = q(t)$. Then, we can rewrite it as

$$q(t) = \sum_{i=0}^{L_2-1} q_i\psi_i(t). \tag{71}$$

Since

$$\sum_{j=0}^{L_1-1} \varphi_j(x) = 1, \tag{72}$$

then

$$q(t) = \sum_{i=0}^{L_1-1}\sum_{j=0}^{L_2-1} q_j\varphi_i(x)\psi_i(t) = \Phi^*(x)Q\Psi(t), \tag{73}$$

where

$$Q = \begin{pmatrix} q_1 & q_2 & \cdots & q_{L_2} \\ q_1 & q_2 & \cdots & q_{L_2} \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_{L_2} \end{pmatrix}. \tag{74}$$

Then,

$$\Phi^*(x)R_I^*VD_t^{2p}\Psi(t) + \Phi^*(x)R_I^*VD_t^{p}\Psi(t) + \Phi^*(x)R_I^*VB\Psi(t) = v_x(x,t) - \Phi^*(x)Q\Psi(t) + \Phi^*(x)R_I^*H\Psi(t). \tag{75}$$

Using the condition $v(0,t) = 0$ and by taking the Riemann integral one more time, we get

$$\begin{aligned} &\Phi^*(x)R_I^*R_I^*VD_t^{2p}\Psi(t) + \Phi^*(x)R_I^*R_I^*VD_t^{p}\Psi(t) + \Phi^*(x)R_I^*R_I^*VB\Psi(t) \\ &\quad = \Phi^*(x)V\Psi(t) - \Phi^*(x)R_I^*Q\Psi(t) + \Phi^*(x)R_I^*R_I^*H\Psi(t). \end{aligned} \tag{76}$$

Since $\{\varphi_j(x)\}_0^{L_1-1}$ are linearly independent, then

$$\begin{aligned}&R_I^* R_I^* V D_t^{2p}\Psi(t) + R_I^* R_I^* V D_t^{p}\Psi(t) + R_I^* R_I^* V B\Psi(t)\\ &\quad = V\Psi(t) - R_I^* Q\Psi(t) + R_I^* R_I^* H\Psi(t).\end{aligned} \tag{77}$$

Now, we need to compute the operational matrix of $I_t^p\Psi$.

Theorem 10. *The operational matrix of the operator $I_t^p\Psi$ is*

$$A_\Psi = \frac{\xi_2^p}{\Gamma(p+2)}\begin{pmatrix} 1 & r_1 & r_2 & \cdots & r_{L_2-2} & r_{L_2-1}\\ 0 & 1 & r_1 & \cdots & r_{L_2-3} & r_{L_2-2}\\ 0 & 0 & 1 & \ddots & r_{L_2-4} & r_{L_2-3}\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & r_1\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}, \tag{78}$$

where $r_s = (s+1)^{p+1} - 2s^{p+1} + (s-1)^{p+1}$, $s = 1, 2, \ldots, L_2 - 1$.

Proof. For any $s \in \{0, 1, \ldots, L_2 - 1\}$, simple calculations imply that

$$I_t^p\psi_s(t) = \frac{1}{\Gamma(p)}\int_0^t (t-v)^{p-1}\psi_s(v)\mathrm{d}v = \begin{cases} 0, & t < s\xi_2,\\ \dfrac{(t-s\xi_2)^p}{\Gamma(p+1)}, & s\xi_2 \le t < (s+1)\xi_2,\\ \dfrac{(t-s\xi_2)^p - (t-s\xi_2-\xi_2)^p}{\Gamma(p+1)}, & (s+1)\xi_2 \le t < T_{\max}.\end{cases} \tag{79}$$

Therefore,

$$I_t^p\psi_s(t) = \sum_{j=0}^{L_2-1} z_{j,s}\psi_j(t), \tag{80}$$

implies that

$$\begin{aligned} z_{j,s} &= \frac{1}{\xi_2}\int_0^{T_{\max}} (I_t^p\psi_s(v))\psi_j(v)\mathrm{d}v\\ &= \frac{1}{\xi_2}\int_{j\xi_2}^{(j+1)\xi_2} (I_t^p\psi_s(v))\mathrm{d}v\\ &= \begin{cases} \dfrac{\xi_2^p}{\Gamma(p+2)}, & 0 \le j = s \le L_2 - 1,\\ \dfrac{\xi_2^p\left((s-j+1)^{p+1} - 2(s-j)^{p+1} + (s-j-1)^{p+1}\right)}{\Gamma(p+2)}, & 0 \le j < s \le L_2 - 1,\\ 0, & 0 \le s < j \le L_2 - 1.\end{cases}\end{aligned} \tag{81}$$

Let $r_s = (s+1)^{p+1} - 2s^{p+1} + (s-1)^{p+1}, s = 1, 2, \ldots, L_2 - 1$. Then, we get the result of the theorem. □

Then, by taking the integral operator I_t^p for both sides of equation (77), we get

$$\begin{aligned}&R_I^* R_I^* V D_t^{p}\Psi(t) - R_I^* R_I^* V D_t^{p}\Psi(0) + R_I^* R_I^* V\Psi(t) - R_I^* R_I^* V\Psi(0) + R_I^* R_I^* V B A_\Psi\Psi(t)\\ &\quad = V A_\Psi\Psi(t) - R_I^* Q A_\Psi\Psi(t) + R_I^* R_I^* H A_\Psi\Psi(t),\end{aligned} \tag{82}$$

which can be simplified using equations (64)-(65) as

$$R_I^* R_I^* V D_t^p \Psi(t) + R_I^* R_I^* V \Psi(t) - R_I^* R_I^* Y + R_I^* R_I^* V B A_\Psi \Psi(t) = V A_\Psi \Psi(t) - R_I^* Q A_\Psi \Psi(t) + R_I^* R_I^* H A_\Psi \Psi(t). \quad (83)$$

Since

$$\sum_{j=0}^{L_2-1} \psi_j(t) = 1, \quad (84)$$

then

$$Y = Y \sum_{j=0}^{L_2-1} \psi_j(t) = Y O_{L_2} \Psi(t), \quad (85)$$

where O_{L_2} is $1 \times L_2$ matrix is given by

$$O_{L_2} = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}. \quad (86)$$

Hence, equation (83) becomes

$$R_I^* R_I^* V D_t^p \Psi(t) + R_I^* R_I^* V \Psi(t) - R_I^* R_I^* Y O_{L_2} \Psi(t) + R_I^* R_I^* V B A_\Psi \Psi(t) = V A_\Psi \Psi(t) - R_I^* Q A_\Psi \Psi(t) + R_I^* R_I^* H A_\Psi \Psi(t). \quad (87)$$

Now, take the integral operator I_t^p for both sides of equation (87) to get

$$R_I^* R_I^* V \Psi(t) - R_I^* R_I^* V \Psi(0) + R_I^* R_I^* V A_\Psi \Psi(t) - R_I^* R_I^* Y O_{L_2} A_\Psi \Psi(t) + R_I^* R_I^* V B A_\Psi A_\Psi \Psi(t) = V A_\Psi A_\Psi \Psi(t) - R_I^* Q A_\Psi A_\Psi \Psi(t) + R_I^* R_I^* H A_\Psi A_\Psi \Psi(t). \quad (88)$$

Using equations (65) and (85), we get

$$R_I^* R_I^* V \Psi(t) - R_I^* R_I^* Y O_{L_2} \Psi(t) + R_I^* R_I^* V A_\Psi \Psi(t) - R_I^* R_I^* Y O_{L_2} A_\Psi \Psi(t) + R_I^* R_I^* V B A_\Psi A_\Psi \Psi(t) = V A_\Psi A_\Psi \Psi(t) - R_I^* Q A_\Psi A_\Psi \Psi(t) + R_I^* R_I^* H A_\Psi A_\Psi \Psi(t). \quad (89)$$

Since $\{\psi_j(t)\}_0^{L_2}$ are linearly independent, then

$$R_I^* R_I^* V - R_I^* R_I^* Y O_{L_2} + R_I^* R_I^* V A_\Psi - R_I^* R_I^* Y O_{L_2} A_\Psi + R_I^* R_I^* V B A_\Psi A_\Psi = V A_\Psi A_\Psi - R_I^* Q A_\Psi A_\Psi + R_I^* R_I^* H A_\Psi A_\Psi. \quad (90)$$

We should note that

$$\nu(\eta, t) = \sum_{i=0}^{L_1-1} \sum_{j=0}^{L_2-1} \nu_{i,j} \varphi_i(\eta) \psi_j(t) = 0, \quad (91)$$

which implies that

$$\Phi(\eta)^* V = 0. \quad (92)$$

Then, we get the following system of algebraic equations:

$$R_I^* R_I^* V - R_I^* R_I^* Y O_{L_2} + R_I^* R_I^* V A_\Psi - R_I^* R_I^* Y O_{L_2} A_\Psi + R_I^* R_I^* V B A_\Psi A_\Psi = V A_\Psi A_\Psi - R_I^* Q A_\Psi A_\Psi + R_I^* R_I^* H A_\Psi A_\Psi, \quad (93)$$

$$\Phi(\eta)^* V = 0. \quad (94)$$

It is worth mentioning that the unknowns are the matrix V of size $L_1 \times L_2$ and the vector $\begin{pmatrix} q_1 & q_2 & \cdots & q_{L_2} \end{pmatrix}$. We use Mathematica to solve the algebraic system (93)-(94).

5. Numerical Results

First, we solve two examples in Theorems 5 and 6, and then, we solve two examples of the general case of problems (13)–(15).

Example 11. Consider the following initial value:

$$D^{(3/2)} u(t) + \alpha D^{(3/4)} u(t) + \beta u(t) = 0, \quad (95)$$

with

$$u(0) = 1, \quad D^{(3/4)} u(0) = 0. \quad (96)$$

Using equation (28), for $\alpha = -5$ and $\beta = 4$, we have

$$\mu_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} = 4, \quad \mu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} = 1. \tag{97}$$

Thus,

$$u(t) = c_1 E_{(3/4)}\left(4t^{(3/4)}\right) + c_2 E_{(3/4)}\left(t^{(3/4)}\right). \tag{98}$$

Using the initial conditions in equation (96) and the fact that $D^p E_p(\mu t^p) = \mu E_p(\mu t^p)$, we have

$$c_1 + c_2 = 1, \quad 4c_1 + c_2 = 0. \tag{99}$$

Thus, $c_1 = (-1/3)$ and $c_2 = (4/3)$ which imply that

$$u(t) = \frac{-1}{3} E_{(3/4)}\left(4t^{(3/4)}\right) + \frac{4}{3} E_{(3/4)}\left(t^{(3/4)}\right). \tag{100}$$

If we choose$\alpha = \beta = 4$, then

$$\mu_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} = -2, \quad \mu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} = -2, \tag{101}$$

which imply that

$$u(t) = c_1 E_{(3/4)}\left(-2t^{(3/4)}\right) + c_2 t^{(3/4)} E_{(3/4),(3/4)}\left(-2t^{(3/4)}\right). \tag{102}$$

The initial conditions give the following system:

$$c_1 = 1, \quad -2c_1 + \frac{3}{4} = 0. \tag{103}$$

Thus, $c_1 = 1$ and $c_2 = (8/3)$. Hence, the solution of problems (95)-(96) *is*

$$u(t) = E_{(3/4)}\left(-2t^{(3/4)}\right) + \frac{8}{3} t^{(3/4)} E_{(3/4),(3/4)}\left(-2t^{(3/4)}\right). \tag{104}$$

Example 12. Consider the following boundary value problem

$$D^{(3/2)} \nu(x,t) - 5\alpha D^{(3/4)} \nu(x,t) + 4\beta \nu(x,t) = \nu_{xx}(x,t), \tag{105}$$

with

$$\nu(0,t) = \nu(1,t) = 0. \tag{106}$$

Using equation (44), we have

$$\nu_j(x) = \sin(j\pi), \quad \lambda_j = -j^2\pi^2, j \in \mathbb{N}. \tag{107}$$

Thus,

$$\mu_{1,j} = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta - 4\lambda_j}}{2} = \frac{5 + \sqrt{9 + 4j^2\pi^2}}{2}, \quad \mu_{2,j} = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta - 4\lambda_j}}{2} = \frac{5 - \sqrt{9 + 4j^2\pi^2}}{2}. \tag{108}$$

Thus,

$$u_j(t) = a_j E_{\frac{3}{4}}\left(\frac{5 + \sqrt{9 + 4j^2\pi^2}}{2} t^{\frac{3}{4}}\right) + c_2 E_{\frac{3}{4}}\left(\frac{5 - \sqrt{9 + 4j^2\pi^2}}{2} t^{\frac{3}{4}}\right). \tag{109}$$

Hence,

$$\nu(x,t) = \sum_{j=0}^{\infty} \nu_j(x) u_j(t) = \sum_{j=0}^{\infty} \left(a_j E_{\frac{3}{4}}\left(\frac{5 + \sqrt{9 + 4j^2\pi^2}}{2} t^{\frac{3}{4}}\right) + c_2 E_{\frac{3}{4}}\left(\frac{5 - \sqrt{9 + 4j^2\pi^2}}{2} t^{\frac{3}{4}}\right)\right) \sin(j\pi). \tag{110}$$

Now, we study the solution of problems (13)–(15) using the OMM numerically.

Example 13. Consider the following problem:

$$D_t^{1.6}\nu(x,t) + D_t^{0.8}\nu(x,t) + t^{0.8}\nu(x,t) = \nu_{xx}(x,t) + h(x,t), \quad (111)$$

$$\nu(0,t) = \nu(1,t) = 0, t \geq 0, \quad (112)$$

$$\nu(x,0) = x^2 - x, D_t^{0.8}\nu(x,0) = 0, 0 \leq x \leq 1, \quad (113)$$

where

$$h(x,t) = (x-1)x\left(t^{2.4} + t^{0.8} + \frac{t^{0.8}\Gamma(13/5)}{\Gamma(9/5)} + \Gamma\left(\frac{13}{5}\right)\right) - 2\left(t^{1.6} + 1\right). \quad (114)$$

Then, the exact solution of problems (111)–(113) is

$$\nu(x,t) = x(x-1)\left(t^{1.6} + 1\right). \quad (115)$$

Let the L_2-error with respect to x is defined by

$$\epsilon(t) = \sqrt{\int_0^1 \left(\nu(x,t) - \nu_{40,40}(x,t)\right)^2 dx}. \quad (116)$$

Then, the error for different values of t is reported in Table 1. In this example, we use $L_1 = L_2 = 40$. The graph of the exact and approximate solutions for $t = 0.3j, j = 0, 1, \ldots, 5$ is given in Figure 1. Note that the exact solution in Figure 1 is given as a solid line while the approximate solution is as dots. Also, the graph of the exact and approximate solutions on the domain $[0,1] \times [0,1.5]$ is given in Figure 2.

Example 14. Consider the following problem:

$$\begin{gathered} D_t^{3/2}\nu(x,t) + D_t^{3/4}\nu(x,t) + t^3\nu(x,t) = \nu_{xx}(x,t) + h(x,t), \\ \nu(0,t) = \nu(1,t) = 0, t \geq 0, \\ \nu(x,0) = x^4 - 2x^2 + x, D_t^{3/4}\nu(x,0) = 0, 0 \leq x \leq 1, \end{gathered} \quad (117)$$

where

$$\begin{aligned} h(x,t) &= t^3x^4E_{3/4}(t) - 2t^3x^2E_{3/4}(t) + t^3xE_{3/4}(t) + 2x^4E_{3/4}(t) - 16x^2E_{3/4}(t) + 2xE_{3/4}(t) \\ &+ 4E_{\frac{3}{4}}(t) - \frac{t^{15/4}x^4}{\Gamma(7/4)} + \frac{12t^{3/4}x^2}{\Gamma(7/4)} + \frac{2t^{15/4}x^2}{\Gamma(7/4)} - \frac{t^{15/4}x}{\Gamma(7/4)} - \frac{4t^{3/4}}{\Gamma(7/4)} - x^4 + 2x^2 - x. \end{aligned} \quad (118)$$

Then, the exact solution of problems (111)–(113) is

$$\nu(x,t) = \left(x^4 - 2x^2 + x\right)\left(E_{\frac{3}{4}}(t) - \frac{t^{3/4}}{\Gamma(7/4)}\right). \quad (119)$$

Then, the error for different values of t is reported in Table 2. In this example, we use $L_1 = L_2 = 45$. The graph of the exact and approximate solutions for $t = 0.3j, j = 0, 1, \ldots, 5$ is given in Figure 3. Note that the exact solution in Figure 3 is given as a solid line while the approximate solution is given as dots. Also, the graph of the exact and approximate solutions on the domain $[0,1] \times [0,1.5]$ is given in Figure 4.

Now, we want to compare our results with [26].

Table 1: The L_2-error for different values of t for Example 13.

t	$\epsilon(t)$
0	0
0.3	1.11×10^{-15}
0.6	1.35×10^{-15}
0.9	3.28×10^{-15}
1.2	6.09×10^{-15}
1.5	8.73×10^{-15}

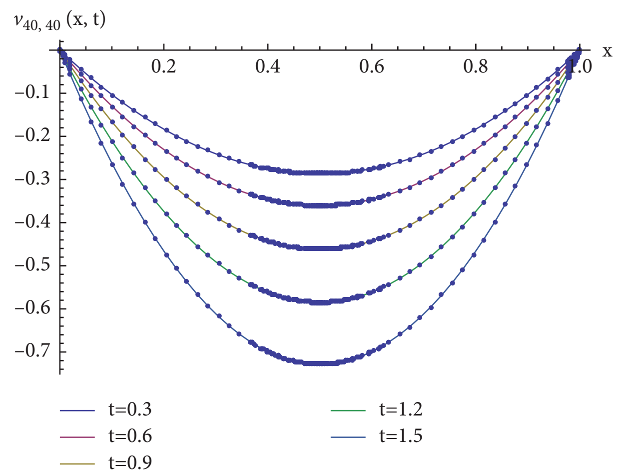

Figure 1: The exact and approximate solutions for $t = 0.3, 0.6, 0.9, 1.2, 1.5$ for Example 13.

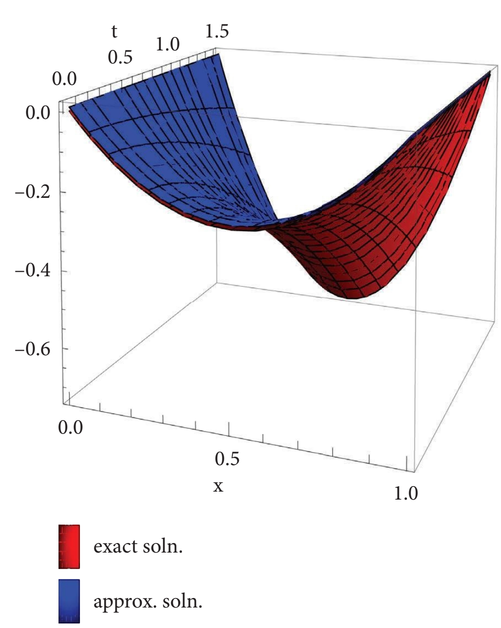

Figure 2: The exact and approximate solutions for Example 13.

Table 2: The L_2-error for different values of t for Example 14.

t	$\epsilon(t)$
0	0
0.3	2.31×10^{-15}
0.6	3.29×10^{-15}
0.9	5.75×10^{-15}
1.2	8.01×10^{-15}
1.5	9.89×10^{-15}

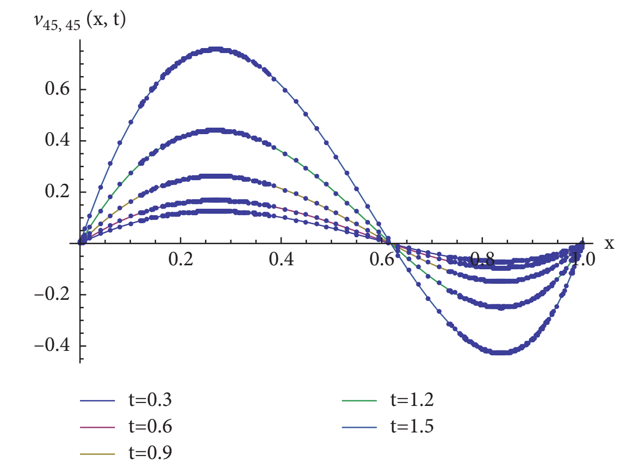

Figure 3: The exact and approximate solutions for $t = 0.3, 0.6, 0.9, 1.2, 1.5$ for Example 14.

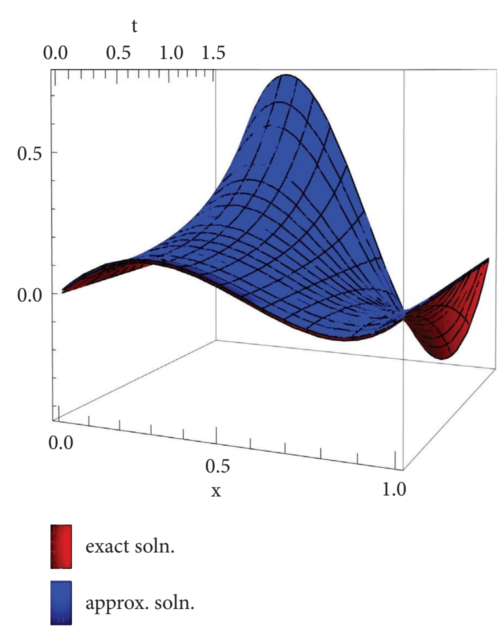

Figure 4: The exact and approximate solutions for Example 14.

Example 15. Consider the following problem:

$$\begin{aligned} D_t^{2p} v(x,t) &= v_{xx}(x,t) + h(x,t), \\ v(0,t) &= v(1,t) = 0, t \geq 0, \\ v(x,0) &= 0, D_t v(x,0) = -\sin(\pi x), 0 \leq x \leq 1, \end{aligned} \tag{120}$$

where

$$h(x,t) = \left(\frac{t^{1-2p}(2t + 2p - 3)}{\Gamma(3-2p)} + \pi^2 \right) \sin(\pi x). \tag{121}$$

Then, the exact solution of problems (111)–(113) is

$$v(x,t) = \sin(\pi x)\left(t^2 - t\right). \tag{122}$$

Table 3: The max-error for different values of p for Example 15.

$2p$	Max-error in our results	Max-error in [26]
1.50	0.00004	$.395 \times 10^{-15}$
1.7	0.00443	8.591×10^{-15}

We will compare our results with [26] based on the maximum error for different values of p. For $p = 0.75$, we take $\Delta x = (1/45), = (1/220)$, and for $p = 0.0.85$, we take $\Delta x = (1/70), = (1/480)$ for the results in [26]. These results are reported in Table 3.

6. Conclusions

In this article, we discuss the differences between the sequential Caputo derivative and the Caputo derivative. We generalize this concept to find the solution of a class of homogeneous fractional wave equations with constant coefficients. Specifically, we derive the analytical solution of a homogeneous second-order fractional equation with constant coefficients in one variable. We provide evidence to support our findings and illustrate their practical application through two examples.

We then shift our focus to a different category of variable coefficient nonhomogeneous fractional wave equations. Since finding exact solutions for these problems can be challenging or even impossible, we employ a numerical technique based on the operational matrix method (OMM) to approximate the solutions. The OMM allows us to reduce the fractional order differential equations to algebraic systems, offering benefits such as low equation setup costs and the absence of projection methods like Galerkin or collocation.

To validate our approach, we provide two examples and compare the exact and approximate solutions. The results demonstrate the accuracy and effectiveness of our strategy, as the graphs of the solutions closely match and the L_2-errors are nearly zero for various values of t.

From these examples, we note the following observations: Example 11 presents the exact solutions of the homogeneous second-order fractional equation with constant coefficients for different choices of α and β based on Theorem 5. Example 12 provides the exact solution of the homogeneous fractional wave equation with constant coefficients based on Theorem 6. Examples 13 and 14 demonstrate the small L_2-errors within $O(10^{-15})$, as shown in Tables 1 and 2, respectively. The approximate solution converges to the exact solution, as depicted in Figures 1 and 4. Figures 1 and 3 display the exact and approximate solutions for different values of t, illustrating their coincidence. The operational matrix method (OMM) proves to be an efficient tool for solving such problems and can be generalized to other problems in physics and engineering. For future work, we plan to extend our study to investigate other types of problems, including diffusion and Laplacian equations. Additionally, we aim to utilize the sequential Caputo derivative to solve systems of fractional differential equations with constant coefficients.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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