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APPROXIMATION BY NÖRLUND TYPE MEANS IN THE GRAND LEBESGUE SPACES WITH VARIABLE EXPONENT

In the present paper the approximation of functions by Nörlund type means in the generalized grand Lebesgue spaces with variable exponent is studied.

Keywords: grand variable exponent Lebesgue spaces, modulus of smoothness, Lipschitz classes, trigonometric approximation, Nörlund means.

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§ 1. Introduction and auxiliary results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. We denote by $L^p(\mathbb{T})$, $1 \leq p < \infty$, the Lebesgue space of all measurable 2π -periodic functions, for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p dx \right)^{1/p} < \infty.$$

Let us denote by $\wp$ the class of Lebesgue measurable functions $p(\cdot): \mathbb{T} \rightarrow [0, \infty)$ such that $1 \leq p_* := \operatorname{ess\,inf}_{x \in \mathbb{T}} p(x) \leq \operatorname{ess\,sup}_{x \in \mathbb{T}} p(x) =: p^* < \infty$. The conjugate exponent of $p(x)$ is shown by

$p'(x) := \frac{p(x)}{p(x) - 1}$. For $p \in \wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π -periodic measurable functions $f: \mathbb{T} \rightarrow \mathbb{C}$ satisfying the condition

$$\int_{\mathbb{T}} |f(x)|^{p(x)} dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0: \int_{\mathbb{T}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \quad (1.1)$$

The class $L^{p(\cdot)}(\mathbb{T})$ with the norm (1.1) is called the *Lebesgue space with variable exponent*. Information on the properties of this space can be found in [11, 27, 28, 44, 45]. We say that a variable exponent $p(x)$ satisfies the *local log-continuity condition*, if there is a positive constant c such that

$$|p(x) - p(y)| \ln \left(\frac{1}{\log |x - y|} \right) \leq c, \quad (1.2)$$

for all $x, y \in [0, 2\pi]$, $|x - y| \leq \frac{1}{2}$, $x \neq y$.

We denote by $\wp^{\log}(\mathbb{T})$ the class of 2π -periodic functions satisfying the condition (1.2). We also define $\wp_0(\mathbb{T}) := \{p(\cdot) \in \wp^{\log}(\mathbb{T}): 1 < p_*\}$.

Let $\theta \geq 0$ and $p \in \wp_0(\mathbb{T})$. We denote by the *generalized grand Lebesgue space with variable exponent* $L^{p(\cdot), \theta}$ the class of all 2π -periodic measurable functions f such that

$$\|f\|_{p(\cdot), \theta} = \sup_{0 < \varepsilon < p_* - 1} \varepsilon^{\frac{\theta}{p_* - \varepsilon}} \|f\|_{p(\cdot) - \varepsilon} < \infty.$$

The space $L^{p(\cdot),\theta}$ was introduced in [30]. Note that when p is a constant and $\theta > 0$, these spaces coincide with the grand Lebesgue spaces introduced by Iwaniec and Sbordone in [20] (for $\theta = 1$) and by Greco, Iwaniec and Sbordone in [16] (for $\theta > 1$). If $p \in \wp$, the embeddings

$$L^{p(\cdot)} \subset L^{p(\cdot),\theta} \subset L^{p(\cdot)-\varepsilon}, \quad 0 < \varepsilon < p_* - 1,$$

hold.

Note [50] that the generalized grand Lebesgue space with variable exponent $L^{p(\cdot),\theta}$ has important applications in different areas of mathematics, physics and mechanics. In particular, the variable exponent Lebesgue spaces have considerable applications in fluid dynamic, especially, for modeling of electrorheological fluids; the grand and generalized grand Lebesgue spaces have been applied in various fields, in particular, in the theory of PDE [21, 42, 43], they are right spaces for the investigations of some nonlinear equations. There are sufficient investigations, relating the fundamental problems of these spaces in view of potential theory, maximal and singular operator theory, where the analogues of the classical results existing in the classical Lebesgue spaces were studied. The detailed information about these investigations can be found in the monographs [10, 11, 31, 32, 46].

Note that the closure of the space $L^{p(\cdot)}(\mathbb{T})$ in $L^{p(\cdot),\theta}(\mathbb{T})$, $\theta > 0$, does not coincide with $L^{p(\cdot),\theta}(\mathbb{T})$ [30]. We denote this closure by $L_*^{p(\cdot),\theta}(\mathbb{T})$. This space is a subspace of $L^{p(\cdot),\theta}(\mathbb{T})$. According to [28] for the functions belonging to this space

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{p_*-1}} \|f\|_{p(\cdot)-\varepsilon} = 0$$

holds.

We suppose that $p(\cdot) \in \wp_0(\mathbb{T})$ and $\theta > 0$. For $f \in L^{p(\cdot),\theta}(\mathbb{T})$, we set [50]

$$(\nu_h f)(x) := \frac{1}{h} \int_0^h f(x+t) dt, \quad 0 < h < \pi, \quad x \in \mathbb{T}.$$

If $p(\cdot) \in \wp_0(\mathbb{T})$, $\theta > 0$, and $f \in L^{p(\cdot),\theta}(\mathbb{T})$, then the shift operator ν_{h_i} is a bounded linear operator on $L^{p(\cdot),\theta}(\mathbb{T})$ [50]:

$$\|\nu_{h_i}(f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leq c_1 \|f\|_{L^{p(\cdot),\theta}(\mathbb{T})}.$$

Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $f \in L^{p(\cdot),\theta}(\mathbb{T})$. The function

$$\Omega_{p(\cdot),\theta}(f, \delta) := \sup_{0 < h \leq \delta} \|f - (\nu_h f)\|_{L^{p(\cdot),\theta}(\mathbb{T})}, \quad \delta > 0,$$

is called the *modulus of continuity* of $f \in L^{p(\cdot),\theta}(\mathbb{T})$.

It can easily be shown that $\Omega_{p(\cdot),\theta}(f, \cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta \rightarrow 0} \Omega_{p(\cdot),\theta}(f, \delta) = 0, \quad \Omega_{p(\cdot),\theta}(f + g, \delta) \leq \Omega_{p(\cdot),\theta}(f, \delta) + \Omega_{p(\cdot),\theta}(g, \delta), \quad \delta > 0,$$

for $f, g \in L^{p(\cdot),\theta}(\mathbb{T})$. Note that a detailed information about properties of the modulus of continuity $\Omega_{p(\cdot),\theta}(f, \cdot)$ can be found in the paper [50].

We use the constants $c, c_1, c_2, \dots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest. We also will use the relation $f = O(g)$ which means that $f \leq cg$ for a constant c independent of f and g . Let $0 < \alpha \leq 1$. The set of functions $f \in L_*^{p(\cdot),\theta}(\mathbb{T})$ such that

$$\Omega_{p(\cdot),\theta}(f, \delta) = O(\delta^\alpha), \quad \delta > 0,$$

is called the *Lipschitz class* $\text{Lip}(\alpha, p(\cdot), \theta)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} Q_k(x, f) \quad (1.3)$$

be the Fourier series of the function $f \in L_1(T)$, where $Q_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx)$, $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$. The n -th *partial sum* of the series (1.3) is defined by

$$S_n(x, f) = \sum_{k=0}^n Q_k(x, f),$$

where

$$Q_0(x, f) := \frac{a_0}{2}; \quad Q_k(x, f) := (a_k(f) \cos kx + b_k(f) \sin kx), \quad k = 1, 2, \dots$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. The sequence $\{p_n\}_0^\infty$ is called *almost monotone decreasing (increasing)*, denoted by $\{p_n\}_0^\infty \in \text{AMDS}$ ($\{p_n\}_0^\infty \in \text{AMIS}$), if there exists a constant c , depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the following inequality holds:

$$p_n \leq cp_m \quad (p_m \leq cp_n),$$

In the proof of the main result we will use the notations

$$\Delta\beta_n := \beta_n - \beta_{n+1}, \quad \Delta_m\beta(n, m) := \beta(n, m) - \beta(n, m+1).$$

As in [38] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as a range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$. Following [5] and [40], the Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad n = 1, 2, \dots,$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, the C_λ -method yields a subsequence of the Cesàro method C_1 , and hence it is regular for any λ . C_λ is obtained by deleting a set of rows from Cesàro matrix.

We suppose that $\{p_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (1.3) as

$$N_n^\lambda(x, f) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} S_m(x, f),$$

where $P_{\lambda(n)} := \sum_{m=0}^{\lambda(n)} p_m \neq 0$, $n \geq 0$, $p_{-1} := P_{-1} := 0$. Note that in the case $p_n = 1$, $n \geq 0$, $N_n^\lambda(x, f)$ is equal to the mean

$$\sigma_n^\lambda(x, f) = \frac{1}{\lambda(n) + 1} \sum_{m=0}^{\lambda(n)} S_m(x, f).$$

Using [33, 46] we introduce two new classes of numerical sequences.

Let $R_{\lambda(n),k} = \frac{1}{(k+1)P_{\lambda(n)}} \sum_{s=\lambda(n)-k}^{\lambda(n)} p_s$. If $(R_{\lambda(n),k}) \in \text{AMDS}$ ($(R_{\lambda(n),k}) \in \text{AMIS}$), then it is said that (p_k) is a λ -almost monotone decreasing (increasing) upper mean sequence, briefly

$(p_k) \in \lambda\text{-AMDUMS}$ ($(p_k) \in \lambda\text{-AMIUMS}$). Note that the classes $\lambda\text{-AMDUMS}$ and $\lambda\text{-AMIUMS}$ are generalizations of the classes AMDUMS and AMIUMS respectively. It is clear that if $\lambda(n) = n$, $n = 1, 2, \dots$, we obtain $\lambda\text{-AMDUMS} = \text{AMDUMS}$ and $\lambda\text{-AMIUMS} = \text{AMIUMS}$ defined in [48].

The best approximation of $f \in L_*^{p(\cdot), \theta}$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{p(\cdot), \theta} := \inf \left\{ \|f - T_n\|_{p(\cdot), \theta} : T_n \in \prod_n \right\}.$$

In the proof of the main result we need the following Lemmas.

Lemma 1.1 (see [50]). *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$. Then for $f \in \text{Lip}(\alpha, p(\cdot), \theta)$, $0 < \alpha \leq 1$, and $n = 1, 2, 3, \dots$ the following estimate holds:*

$$\|f - S_n(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} = O(n^{-\alpha}).$$

Lemma 1.2 (see [50]). *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$. Then for $f \in \text{Lip}(1, p(\cdot), \theta)$ and $n = 1, 2, 3, \dots$ the following estimate holds:*

$$\|S_n(\cdot, f) - \sigma_n(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} = O(n^{-1}).$$

Lemma 1.3 (see [33]). *Let $\{p_n\}$ be a positive sequence. Let following conditions hold:*

- (1) $(p_n) \in \lambda\text{-AMDUMS}$ or
- (2) $(p_n) \in \lambda\text{-AMIUMS}$ and $(\lambda(n) + 1)p_{\lambda(n)} = O(P_{\lambda(n)})$.

Then

$$\Lambda := \sum_{m=0}^{\lambda(n)} \frac{p_{\lambda(n)-m}}{(m+1)^\alpha} = O_\alpha \left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^\alpha} \right).$$

for $0 < \alpha < 1$.

Theorem 1.1 (see [33]). *The following properties are valid:*

- (1) *if $(p_n) \in \text{AMDS}$, then $(p_n) \in \lambda\text{-AMIUMS}$;*
- (2) *if $(p_n) \in \text{AMIS}$, then $(p_n) \in \lambda\text{-AMDUMS}$;*
- (3) *if $\sum_{s=0}^{\lambda(n)-1} \left| \Delta \left(\frac{p_s}{P_{\lambda(n)}} \right) \right| = O((\lambda(n))^{-1})$, then $\sum_{s=0}^{\lambda(n)-1} |\Delta(R_{\lambda(n), s})| = O((\lambda(n))^{-1})$;*
- (4) *if $\sum_{s=1}^{\lambda(n)-1} s \left| \Delta \left(\frac{p_s}{P_{\lambda(n)}} \right) \right| = O(1)$, then $\sum_{s=0}^{\lambda(n)-2} |\Delta(R_{\lambda(n), s})| = O((\lambda(n))^{-1})$.*

§2. Main Results

The problems of approximation theory in variable and grand variable exponent Lebesgue spaces have been investigated by several authors (see, for example, [1–4, 12–15, 17, 22, 26, 29, 47, 50–52, 54, 55]). In the present paper we study the approximation of functions by Nörlund type means in the generalized grand Lebesgue space with variable exponent $L^{p(\cdot), \theta}$, $\theta > 0$. The results obtained in this work are generalization of the results [17, 32] to the generalized grand Lebesgue space with variable exponent. Similar approximation problems in different spaces have been investigated in [6–9, 9, 17–19, 22–26, 33–41, 48–54, 56–58].

Note that, in the proof of the main results we use the method as in the proofs of [17, 33].

Our main results are the following.

Theorem 2.1. Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $\{p_n\}_{n=0}^\infty$ be a sequence of positive real numbers. Also, let the following conditions hold:

$$\{p_n\}_0^\infty \in \lambda\text{-AMDUMS}$$

or

$$\{p_n\}_0^\infty \in \lambda\text{-AMIUMS}, \text{ and } (\lambda(n) + 1)p_{\lambda(n)} = O(P_{\lambda(n)}). \quad (2.1)$$

Then, for $f \in \text{Lip}(\alpha, p(\cdot), \theta)$, $0 < \alpha < 1$, the relation

$$\|f - N_n^\lambda(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} = O((\lambda(n) + 1)^{-\alpha}), \quad n \in \mathbb{N} \cup \{0\},$$

holds.

Theorem 2.2. Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $\{p_n\}_{n=0}^\infty$ be a sequence of positive real numbers. Also, let the following condition holds:

$$\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n), m} - R_{\lambda(n), m+1}| = O((\lambda(n))^{-1}).$$

Then, for $f \in \text{Lip}(1, p(\cdot), \theta)$, the relation

$$\|f - N_n^\lambda(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} = O((\lambda(n))^{-1}), \quad n = 1, 2, \dots$$

holds.

Remark 2.1. Theorem 2.2 gives the same degree of approximation with conditions different from those of Theorem 1.1, considering the case $\alpha = 1$.

Remark 2.2. If $\theta = 0$ and $\lambda(n) = n$, $n = 1, 2, \dots$, then from Theorems 2.1 and 2.2 we obtain results of [17].

Note that, since $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$, $0 < \alpha \leq 1$, the results obtained in [33] give sharper estimates than those of results in [17].

§ 3. Proofs of the main results

Proof of Theorem 2.1. It is clear that

$$N_n^\lambda(x, f) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \{f(x) - S_m(x, f)\}. \quad (3.1)$$

Then using Lemma 1.1 and Lemma 1.3 and (2.1) we have

$$\begin{aligned} \|N_n^\lambda(\cdot, f) - f\|_{L^{p(\cdot), \theta}(\mathbb{T})} &\leq \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \|f - S_m(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} \|f - S_m(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} + \|f - S_0(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} \\ &= \frac{1}{P_{\lambda(n)}} O\left(\sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} (m+1)^{-\alpha}\right) \\ &= \frac{1}{P_{\lambda(n)}} O(P_{\lambda(n)} (\lambda(n) + 1)^{-\alpha}) = O((\lambda(n) + 1)^{-\alpha}). \end{aligned}$$

The proof of the Theorem is completed. □

Proof of Theorem 2.2. We can write the following equality:

$$F_n^\lambda(x, f) := N_n^\lambda(x, f) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \{S_m(x, f) - f(x)\}.$$

Using Abel's transformation, we find that [33]

$$\begin{aligned} F_n^\lambda(x, f) &= \sum_{m=1}^{\lambda(n)-1} (S_m(x, f) - S_{m+1}(x, f)) \frac{1}{P_{\lambda(n)}} \sum_{s=0}^m p_{\lambda(n)-s} + S_{\lambda(n)}(x, f) - f(x) \\ &= - \sum_{m=0}^{\lambda(n)-1} (m+1) Q_{m+1}(x, f) R_{\lambda(n),m} + S_{\lambda(n)}(x, f) - f(x) \\ &= - \sum_{m=0}^{\lambda(n)-2} (R_{\lambda(n),m} - R_{\lambda(n),m+1}) \sum_{s=0}^m (s+1) Q_{s+1}(x, f) \\ &\quad - ((\lambda(n) P_{\lambda(n)})^{-1}) \sum_{s=1}^{\lambda(n)} p_s \sum_{s=0}^{\lambda(n)-1} (s+1) Q_{s+1}(x, f) + S_{\lambda(n)}(x, f) - f(x) \end{aligned} \quad (3.2)$$

Using (3.2), we obtain

$$\begin{aligned} \|F_n^\lambda(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} &\leq \sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| \left\| \sum_{s=1}^{m-1} s Q_s(\cdot, f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\ &\quad + ((\lambda(n))^{-1}) \left\| \sum_{s=1}^{\lambda(n)} s Q_s(\cdot, f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} + \|S_{\lambda(n)}(\cdot, f) - f\|_{L^{p(\cdot),\theta}(\mathbb{T})}. \end{aligned}$$

It is clear that the equality

$$\begin{aligned} \sum_{s=1}^{\lambda(n)} s Q_s(f; x) &= (\lambda(n) + 1) S_{\lambda(n)}(x, f) - \sigma_{\lambda(n)}(x, f) \\ S_n(x, f) - \sigma_n(x, f) &= \frac{1}{n+1} \sum_{k=1}^n k Q_k(x, f). \end{aligned} \quad (3.3)$$

holds. Then, from Lemma 1.2 and (3.3), we have

$$\left\| \sum_{s=1}^{\lambda(n)} s Q_s(\cdot, f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(1). \quad (3.4)$$

Thus, use of Lemma 1.2 and (3.4) gives us

$$\|F_n^\lambda(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O \left(\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| \right) + O((\lambda(n))^{-1}). \quad (3.5)$$

Now we suppose that the condition

$$\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| = O(\lambda(n)^{-1})$$

is satisfied. Then the last relation and (3.5) imply that

$$\|f - N_n^\lambda(\cdot, f)\|_{L^{p(\cdot), \theta}(\mathbb{T})} = O((\lambda(n))^{-1}), \quad n = 1, 2, \dots$$

The proof of Theorem 2.2 is completed. $\square$

REFERENCES

1. Akgün R., Kokilashvili V. On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces, *Banach Journal of Mathematical Analysis*, 2011, vol. 5, no. 1, pp. 70–82. <https://doi.org/10.15352/bjma/1313362981>
2. Akgün R. Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent, *Ukrainian Mathematical Journal*, 2011, vol. 63, no. 1, pp. 1–26. <https://doi.org/10.1007/s11253-011-0485-0>
3. Akgün R., Kokilashvili V. The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces, *Georgian Mathematical Journal*, 2011, vol. 18, issue 3, pp. 399–423. <https://doi.org/10.1515/GMJ.2011.0037>
4. Akgün R., Kokilashvili V. Some approximation problems for (α, ψ) -differentiable functions in weighted variable exponent Lebesgue spaces, *Journal of Mathematical Sciences*, 2012, vol. 186, no. 2, pp. 139–152. <https://doi.org/10.1007/s10958-012-0980-3>
5. Armitage D.H., Maddox I.J. A new type of Cesàro mean, *Analysis*, 1989, vol. 9, issues 1–2, pp. 195–204. <https://doi.org/10.1524/anly.1989.9.12.195>
6. Chandra P. Approximation bu Nörlund operators, *Matematički Vesnik*, 1986, vol. 38, issue 3, pp. 263–269.
7. Chandra P. Functions of classes L_p and $\text{Lip}(\alpha, p)$ their Riesz means, *Rivista di Matematica della Università di Parma. Serie 4*, 1986, vol. 12, pp. 275–282. <http://rivista.math.unipr.it/fulltext/1986-12/1986-12-275.pdf>
8. Chandra P. A note on degree of approximation by Nörlund and Riesz operators, *Matematički Vesnik*, 1990, vol. 42, issue 1, pp. 9–10.
9. Chandra P. Trigonometric approximation of functions in L_p -norm, *Journal of Mathematical Analysis and Applications*, 2002, vol. 275, issue 1, pp. 13–26. [https://doi.org/10.1016/S0022-247X\(02\)00211-1](https://doi.org/10.1016/S0022-247X(02)00211-1)
10. Cruz-Uribe D.V., Fiorenza A. *Variable Lebesgue spaces. Foundation and harmonic analysis*, Basel: Birkhäuser, 2013. <https://doi.org/10.1007/978-3-0348-0548-3>
11. Diening L., Harjulehto P., Hästö P., Růžička M. *Lebesgue and Sobolev spaces with variable exponents*, Heidelberg: Springer, 2011. <https://doi.org/10.1007/978-3-642-18363-8>
12. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces, *Bulletin of the Georgian National Academy of Sciences*, 2013, vol. 7, no. 1, pp. 11–15. <http://science.org.ge/old/moambe/7-1/Danelia%2011-15.pdf>
13. Danelia N., Kokilashvili V. Approximation of periodic functions in grand variable exponent Lebesgue spaces, *Proceedings of A. Razmadze Mathematical Institute*, 2014, vol. 164, pp. 100–103. <https://zbmath.org/1296.41004>
14. Danelia V., Kokilashvili V., Tsanava Ts. Some approximation results in subspace of weighted grand Lebesgue spaces, *Proceedings of A. Razmadze Mathematical Institute*, 2014, vol. 164, pp. 104–108. <https://zbmath.org/1297.42003>
15. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces, *Georgian Mathematical Journal*, 2016, vol. 23, issue 1, pp. 43–53. <https://doi.org/10.1515/gmj-2015-0059>
16. Sbordone C., Greco L., Iwaniec T. Inverting the p -harmonic operators, *Manuscripta Mathematica*, 1997, vol. 92, issue 2, pp. 249–258. <https://eudml.org/doc/156263>
17. Guven A., Israfilov D.M. Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$, *Journal of Mathematical Inequalities*, 2010, vol. 4, no. 2, pp. 285–299. <https://doi.org/10.7153/jmi-04-25>

18. Güven A., Israfilov D.M. Approximation by means of Fourier trigonometric series in weighted Orlicz spaces, *Advanced Studies in Contemporary Mathematics (Kyungshang)*, 2009, vol. 19, issue 2, pp. 283–295.
19. Il'yasov N.A. Approximation of periodic functions by Zygmund means, *Mathematical Notes of the Academy of Sciences of the USSR*, 1986, vol. 39, issue 3, pp. 200–209.
<https://doi.org/10.1007/BF01170248>
20. Iwaniec T., Sbordone C. On the integrability of the Jacobian under minimal hypotheses, *Archive for Rational Mechanics and Analysis*, 1992, vol. 119, no. 2, pp. 129–143.
<https://doi.org/10.1007/BF00375119>
21. Iwaniec T., Sbordone C. Riesz transform and elliptic PDEs with VMO coefficients, *Journal d'Analyse Mathématique*, 1998, vol. 74, no. 1, pp. 183–212. <https://doi.org/10.1007/BF02819450>
22. Jafarov S.Z. Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents, *Ukrainian Mathematical Journal*, 2015, vol. 66, no. 10, pp. 1509–1518. <https://doi.org/10.1007/s11253-015-1027-y>
23. Jafarov S.Z. Linear methods of summing Fourier series and approximation in weighted Orlicz spaces, *Turkish Journal of Mathematics*, 2018, vol. 42, no. 6, article 6, pp. 2916–2925.
<https://doi.org/10.3906/mat-1804-31>
24. Jafarov S.Z. Approximation by linear means of Fourier series in weighted Orlicz spaces, *Proceedings of the Institute of Mathematics and Mechanics*, 2017, vol. 43, no. 2, pp. 175–187.
<https://proc.imm.az/volumes/43-2/43-02-01.pdf>
25. Jafarov S.Z. On approximation of a weighted Lipschitz class functions by means $t_n(f; x)$, $N_n^\beta(f; x)$ and $R_n^\beta(f, x)$ of Fourier series, *Transactions Issue Mathematics. National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences*, 2020, vol. 40, no. 4, pp. 118–129.
<https://doi.org/10.29228/proc.50>
26. Jafarov S.Z. On approximation of functions by means of Fourier trigonometric series in weighted generalized grand Lebesgue spaces, *Advanced Studies: Euro-Tbilisi Mathematical Journal*, 2022, special issue 10, pp. 277–291.
[https://tcms.org.ge/Journals/ASETMJ/Special issue/10/PDF/asetmj_SpIssue_10_20.pdf](https://tcms.org.ge/Journals/ASETMJ/Special%20issue/10/PDF/asetmj_SpIssue_10_20.pdf)
27. Kováčik O., Rákosník J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Mathematical Journal*, 1991, vol. 41, issue 4, pp. 592–618. <https://doi.org/10.21136/CMJ.1991.102493>
28. Kokilashvili V. On a progress in the theory of integral operators in weighted Banach function spaces, *Function Spaces, Differential Operators and Nonlinear Analysis: Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, 2004*, Praha: Mathematical Institute of the Academy of Sciences of the Czech Republic, 2005, pp. 152–174.
29. Kokilashvili V.M., Samko S.G. Operators of harmonic analysis in weighted spaces with non-standard growth, *Journal of Mathematical Analysis and Applications*, 2009, vol. 352, issue 1, pp. 15–34.
<https://doi.org/10.1016/j.jmaa.2008.06.056>
30. Kokilashvili V., Meskhi A. Maximal and Calderón–Zygmund operators in grand variable exponent Lebesgue spaces, *Georgian Mathematical Journal*, 2014, vol. 21, issue 4, pp. 447–461.
<https://doi.org/10.1515/gmj-2014-0047>
31. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. *Integral operators in non-standard function spaces. Volume 1: Variable exponent Lebesgue and amalgam spaces*, Cham: Birkhäuser, 2016.
<https://doi.org/10.1007/978-3-319-21015-5>
32. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. *Integral operators in non-standard function spaces. Volume 2: Variable exponent Hölder, Morrey–Campanato and grand spaces*, Cham: Birkhäuser, 2016.
<https://doi.org/10.1007/978-3-319-21018-6>
33. Krasniqi X.Z. Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$, *Palestine Journal of Mathematics*, 2017, vol. 6, no. 1, pp. 84–93.
<https://zbmath.org/1352.42004>

34. Krasniqi X.Z. Approximation of periodic functions by sub-matrix means of their Fourier series, *TWMS Journal of Applied and Engineering Mathematics*, 2020, vol. 10, no. 1, pp. 279–287. <http://jaem.isikun.edu.tr/web/index.php/archive/104-vol10no1/513-approximation-of-periodic-functions-by-sub-matrix-means-of-their-fourier-series>
35. Krasniqi X.Z. On degree of approximation of continuous functions by a linear transformation of their Fourier series, *Communications in Mathematics*, 2022, vol. 30, issue 1, pp. 37–46. <https://doi.org/10.46298/cm.9273>
36. Krasniqi X.Z., Lenski W., Szal B. Seminormed approximation by deferred matrix means of integrable functions in $H_P^{(\omega)}$ space, *Results in Mathematics*, 2022, vol. 77, issue 4, article number: 145. <https://doi.org/10.1007/s00025-022-01696-3>
37. Leindler L. Trigonometric approximation in L_p -norm, *Journal of Mathematical Analysis and Applications*, 2005, vol. 302, issue 1, pp. 129–136. <https://doi.org/10.1016/J.JMAA.2004.07.049>
38. Mittal M.L., Mradul Veer Singh. Approximation of signals (functions) by trigonometric polynomials in L_p -norm, *International Journal of Mathematics and Mathematical Sciences*, 2014, vol. 2014, article ID: 267383. <https://doi.org/10.1155/2014/267383>
39. Mohaparta R.N., Russell D.C. Some direct and inverse theorems in approximation of functions, *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics*, 1983, vol. 34, issue 2, pp. 143–154. <https://doi.org/10.1017/S144678870002317X>
40. Osikiewicz J.A. Equivalence results for Cesàro submethods, *Analysis*, 2000, vol. 20, issue 1, pp. 35–43. <https://doi.org/10.1524/anly.2000.20.1.35>
41. Quade E.S. Trigonometric approximation in the mean, *Duke Mathematical Journal*, 1937, vol. 3, no. 3, pp. 529–543. <https://doi.org/10.1215/S0012-7094-37-00342-9>
42. Sbordon C. Grand Sobolev spaces and their applications to variational problems, *Le Matematiche*, 1996, vol. 51, no. 2, pp. 335–347. <https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/443>
43. Sbordon C. Nonlinear elliptic equations with right hand side in nonstandard spaces, *Atti del Seminario Matematico e Fisico dell'Università di Modena*, 1998, vol. 46 suppl., pp. 361–368. <https://zbmath.org/0913.35050>
44. Samko S.G. Convolution type operators in $L^{p(x)}$, *Integral Transforms and Special Functions*, 1998, vol. 7, issues 1–2, pp. 123–144. <https://doi.org/10.1080/10652469808819191>
45. Sharapudinov I.I. The topology of the space $\mathcal{L}^{p(t)}([0, 1])$, *Mathematical Notes of the Academy of Sciences of the USSR*, 1979, vol. 26, issue 4, pp. 796–806. <https://doi.org/10.1007/BF01159546>
46. Sharapudinov I.I. *Nekotorye voprosy teorii priblizhenii v prostranstvakh Lebeगा s peremennym pokazatelem* (Some questions of approximation theory in the Lebesgue spaces with variable exponent), Vladikavkaz: Southern Mathematical Institute of the Vladikavkaz Scientific Center of the Russian Academy of Sciences and the Government of the Republic of North Ossetia–Alania, 2012. <https://www.elibrary.ru/item.asp?id=22887342>
47. Sharapudinov I.I. Approximation of functions in variable-exponent Lebesgue and Sobolev spaces by finite Fourier–Haar series, *Sbornik: Mathematics*, 2014, vol. 205, issue 2, pp. 291–306. <https://doi.org/10.1070/SM2014v205n02ABEH004376>
48. Szal B. Trigonometric approximation by Nörlund type means in L^p -norm, *Commentationes Mathematicae Universitatis Carolinae*, 2009, vol. 50, issue 4, pp. 575–589. <http://dml.cz/dmlcz/137448>
49. Sonker S., Singh U. Approximation of signals (functions) belonging to $\text{Lip}(\alpha, p, \omega)$ -class using trigonometric polynomials, *Procedia Engineering*, 2012, vol. 38, pp. 1575–1585. <https://doi.org/10.1016/j.proeng.2012.06.193>
50. Testici A., Israfilov D.M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent, *Applicable Analysis*, 2021, vol. 100, issue 4, pp. 819–834. <https://doi.org/10.1080/00036811.2019.1622680>
51. Testici A. Approximation by Nörlund and Riesz means in weighted Lebesgue spaces with variable exponent, *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*, 2019, vol. 68, no. 2, pp. 2014–2025. <https://doi.org/10.31801/cfsuasmas.460449>

52. Testici A., Israfilzade D. M. Linear methods of approximation in weighted Lebesgue spaces with variable exponent, *Hacettepe Journal of Mathematics and Statistics*, 2021, vol. 50, issue 3, pp. 744–753. <https://doi.org/10.15672/hujms.798028>
53. Testici A., Israfilov D. M. Approximation by matrix transforms in Morrey spaces, *Problemy Analiza — Issues of Analysis*, 2021, vol. 10 (28), issue 2, pp. 79–98. <https://doi.org/10.15393/j3.art.2021.9635>
54. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent, *Journal of Numerical Analysis and Approximation Theory*, 2021, vol. 50, no. 1, pp. 60–72. <https://doi.org/10.33993/jnaat501-1234>
55. Volosivets S. S. Approximation of functions and their conjugate in variable Lebesgue spaces, *Sbornik: Mathematics*, 2017, vol. 208, issue 1, pp. 44–59. <https://doi.org/10.1070/SM8636>
56. Volosivets S. S. Modified modulus of smoothness and approximation in weighted Lorentz spaces by Borel and Euler means, *Problemy Analiza — Issues of Analysis*, 2021, vol. 10 (28), issue 1, pp. 87–100. <https://doi.org/10.15393/j3.art.2021.8950>
57. Volosivets S. S. Approximation by Vilenkin polynomials in weighted Orlicz spaces, *Analysis Mathematica*, 2021, vol. 47, issue 2, pp. 437–449. <https://doi.org/10.1007/s10476-021-0086-6>
58. Zygmund A. *Trigonometric series*, Cambridge: Cambridge University Press, 2003. <https://doi.org/10.1017/CBO9781316036587>

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Аппроксимация средними Нёрлунда в гранд-пространствах Лебега с переменным показателем

Ключевые слова: гранд-пространства Лебега с переменным показателем, модуль гладкости, классы Липшица, тригонометрическая аппроксимация, средние Нёрлунда.

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В настоящей работе исследуется аппроксимация функций средними Нёрлунда в обобщенных гранд-пространствах Лебега с переменным показателем.

СПИСОК ЛИТЕРАТУРЫ

1. Akgün R., Kokilashvili V. On converse theorems of trigonometric approximation in weighted variable exponent Lebesgue spaces // Banach Journal of Mathematical Analysis. 2011. Vol. 5. No. 1. P. 70–82. <https://doi.org/10.15352/bjma/1313362981>
2. Akgün R. Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent // Ukrainian Mathematical Journal. 2011. Vol. 63. No. 1. P. 1–26. <https://doi.org/10.1007/s11253-011-0485-0>
3. Akgün R., Kokilashvili V. The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces // Georgian Mathematical Journal. 2011. Vol. 18. Issue 3. P. 399–423. <https://doi.org/10.1515/GMJ.2011.0037>
4. Akgün R., Kokilashvili V. Some approximation problems for (α, ψ) -differentiable functions in weighted variable exponent Lebesgue spaces // Journal of Mathematical Sciences. 2012. Vol. 186. No. 2. P. 139–152. <https://doi.org/10.1007/s10958-012-0980-3>
5. Armitage D.H., Maddox I.J. A new type of Cesàro mean // Analysis. 1989. Vol. 9. Issues 1–2. P. 195–204. <https://doi.org/10.1524/anly.1989.9.12.195>
6. Chandra P. Approximation bu Nörlund operators // Matematički Vesnik. 1986. Vol. 38. Issue 3. P. 263–269.
7. Chandra P. Functions of classes L_p and $Lip(\alpha, p)$ their Riesz means // Rivista di Matematica della Università di Parma. Serie 4. 1986. Vol. 12. P. 275–282. <http://rivista.math.unipr.it/fulltext/1986-12/1986-12-275.pdf>
8. Chandra P. A note on degree of approximation by Nörlund and Riesz operators // Matematički Vesnik. 1990. Vol. 42. Issue 1. P. 9–10.
9. Chandra P. Trigonometric approximation of functions in L_p -norm // Journal of Mathematical Analysis and Applications. 2002. Vol. 275. Issue 1. P. 13–26. [https://doi.org/10.1016/S0022-247X\(02\)00211-1](https://doi.org/10.1016/S0022-247X(02)00211-1)
10. Cruz-Uribe D. V., Fiorenza A. Variable Lebesgue spaces. Foundations and harmonic analysis. Basel: Birkhäuser, 2013. <https://doi.org/10.1007/978-3-0348-0548-3>
11. Diening L., Harjulehto P., Hästö P., Růžička M. Lebesgue and Sobolev spaces with variable exponents. Heidelberg: Springer, 2011. <https://doi.org/10.1007/978-3-642-18363-8>
12. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue spaces // Bulletin of the Georgian National Academy of Sciences. 2013. Vol. 7. No. 1. P. 11–15. <http://science.org.ge/old/moambe/7-1/Danelia%2011-15.pdf>
13. Danelia N., Kokilashvili V. Approximation of periodic functions in grand variable exponent Lebesgue spaces // Proceedings of A. Razmadze Mathematical Institute. 2014. Vol. 164. P. 100–103. <https://zbmath.org/1296.41004>
14. Danelia V., Kokilashvili V., Tsanova Ts. Some approximation results in subspace of weighted grand Lebesgue spaces // Proceedings of A. Razmadze Mathematical Institute. 2014. Vol. 164. P. 104–108. <https://zbmath.org/1297.42003>

15. Danelia N., Kokilashvili V. Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces // Georgian Mathematical Journal. 2016. Vol. 23. Issue 1. P. 43–53. <https://doi.org/10.1515/gmj-2015-0059>
16. Sbordonc C., Greco L., Iwaniec T. Inverting the p -harmonic operators // Manuscripta Mathematica. 1997. Vol. 92. Issue 2. P. 249–258. <https://eudml.org/doc/156263>
17. Guven A., Israfilov D.M. Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$ // Journal of Mathematical Inequalities. 2010. Vol. 4. No. 2. P. 285–290. <https://doi.org/10.7153/jmi-04-25>
18. Güven A., Israfilov D. M. Approximation by means of Fourier trigonometric series in weighted Orlicz spaces // Advanced Studies in Contemporary Mathematics (Kyungshang). 2009. Vol. 19. Issue 2. P. 283–295.
19. Ильясов Н. А. Приближение периодических функций средними Зигмунда // Математические заметки. 1986. Т. 39. Вып. 3. С. 367–382. <https://www.mathnet.ru/rus/mzm5056>
20. Iwaniec T., Sbordonc C. On the integrability of the Jacobian under minimal hypotheses // Archive for Rational Mechanics and Analysis. 1992. Vol. 119. No. 2. P. 129–143. <https://doi.org/10.1007/BF00375119>
21. Iwaniec T., Sbordonc C. Riesz transform and elliptic PDEs with VMO coefficients // Journal d'Analyse Mathématique. 1998. Vol. 74. No. 1. P. 183–212. <https://doi.org/10.1007/BF02819450>
22. Jafarov S.Z. Linear methods for summing Fourier series and approximation in weighted Lebesgue spaces with variable exponents // Украинский математический журнал. 2014. Т. 66. № 10. С. 1348–1356. <https://umj.imath.kiev.ua/index.php/umj/article/view/2226>
23. Jafarov S.Z. Linear methods of summing Fourier series and approximation in weighted Orlicz spaces // Turkish Journal of Mathematics. 2018. Vol. 42. No. 6. Article 6. P. 2916–2925. <https://doi.org/10.3906/mat-1804-31>
24. Jafarov S.Z. Approximation by linear means of Fourier series in weighted Orlicz spaces // Proceedings of the Institute of Mathematics and Mechanics. National Academy of Sciences of Azerbaijan. 2017. Vol. 43. No. 2. P. 175–187. <https://proc.imm.az/volumes/43-2/43-02-01.pdf>
25. Jafarov S.Z. On approximation of a weighted Lipschitz class functions by means $t_n(f; x)$, $N_n^\beta(f; x)$ and $R_n^\beta(f, x)$ of Fourier series // Transactions Issue Mathematics. National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences. 2020. Vol. 40. No. 4. P. 118–129. <https://doi.org/10.29228/proc.50>
26. Jafarov S.Z. On approximation of functions by means of Fourier trigonometric series in weighted generalized grand Lebesgue spaces // Advanced Studies: Euro-Tbilisi Mathematical Journal. 2022. Special Issue 10. P. 277–291. [https://tcms.org.ge/Journals/ASETMJ/Special issue/10/PDF/asetmj_SpIssue_10_20.pdf](https://tcms.org.ge/Journals/ASETMJ/Special%20issue/10/PDF/asetmj_SpIssue_10_20.pdf)
27. Kováčik O., Rákosník J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$ // Czechoslovak Mathematical Journal. 1991. Vol. 41. Issue 4. P. 592–618. <https://doi.org/10.21136/CMJ.1991.102493>
28. Kokilashvili V. On a progress in the theory of integral operators in weighted Banach function spaces // Function Spaces, Differential Operators and Nonlinear Analysis: Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, 2004. Praha: Mathematical Institute of the Academy of Sciences of the Czech Republic, 2005. P. 152–174.
29. Kokilashvili V.M., Samko S.G. Operators of harmonic analysis in weighted spaces with non-standard growth // Journal of Mathematical Analysis and Applications. 2009. Vol. 352. Issue 1. P. 15–34. <https://doi.org/10.1016/j.jmaa.2008.06.056>
30. Kokilashvili V., Meskhi A. Maximal and Calderón–Zygmund operators in grand variable exponent Lebesgue spaces // Georgian Mathematical Journal. 2014. Vol. 21. Issue 4. P. 447–461. <https://doi.org/10.1515/gmj-2014-0047>
31. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. Integral operators in non-standard function spaces. Volume 1: Variable exponent Lebesgue and amalgam spaces. Cham: Birkhäuser, 2016. <https://doi.org/10.1007/978-3-319-21015-5>

32. Kokilashvili V., Meskhi A., Rafeiro H., Samko S. Integral operators in non-standard function spaces. Volume 2: Variable exponent Hölder, Morrey–Campanato and grand spaces. Cham: Birkhäuser, 2016. <https://doi.org/10.1007/978-3-319-21018-6>
33. Krasniqi X.Z. Trigonometric approximation of (signals) functions by Nörlund type means in the variable space $L^{p(x)}$ // Palestine Journal of Mathematics. 2017. Vol. 6. No. 1. P. 84–93. <https://zbmath.org/1352.42004>
34. Krasniqi X.Z. Approximation of periodic functions by sub-matrix means of their Fourier series // TWMS Journal of Applied and Engineering Mathematics. 2020. Vol. 10. No. 1. P. 279–287. <http://jaem.isikun.edu.tr/web/index.php/archive/104-vol10no1/513-approximation-of-periodic-functions-by-sub-matrix-means-of-their-fourier-series>
35. Krasniqi X.Z. On degree of approximation of continuous functions by a linear transformation of their Fourier series // Communications in Mathematics. 2022. Vol. 30. Issue 1. P. 37–46. <https://doi.org/10.46298/cm.9273>
36. Krasniqi X.Z., Lenski W., Szal B. Seminormed approximation by deferred matrix means of integrable functions in $H_P^{(\omega)}$ space // Results in Mathematics. 2022. Vol. 77. Issue 4. Article number: 145. <https://doi.org/10.1007/s00025-022-01696-3>
37. Leindler L. Trigonometric approximation in L_p -norm // Journal of Mathematical Analysis and Applications. 2005. Vol. 302. Issue 1. P. 129–136. <https://doi.org/10.1016/J.JMAA.2004.07.049>
38. Mittal M.L., Mradul Veer Singh. Approximation of signals (functions) by trigonometric polynomials in L_p -norm // International Journal of Mathematics and Mathematical Sciences. 2014. Vol. 2014. Article ID: 267383. <https://doi.org/10.1155/2014/267383>
39. Mohaparta R.N., Russell D.C. Some direct and inverse theorems in approximation of functions // Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics. 1983. Vol. 34. Issue 2. P. 143–154. <https://doi.org/10.1017/S144678870002317X>
40. Osikiewicz J.A. Equivalence results for Cesàro submethods // Analysis. 2000. Vol. 20. Issue 1. P. 35–43. <https://doi.org/10.1524/anly.2000.20.1.35>
41. Quade E.S. Trigonometric approximation in the mean // Duke Mathematical Journal. 1937. Vol. 3. No. 3. P. 529–543. <https://doi.org/10.1215/S0012-7094-37-00342-9>
42. Sbordon C. Grand Sobolev spaces and their applications to variational problems // Le Matematiche. 1996. Vol. 51. No. 2. P. 335–347. <https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/443>
43. Sbordon C. Nonlinear elliptic equations with right hand side in nonstandard spaces // Atti del Seminario Matematico e Fisico dell'Università di Modena. 1998. Vol. 46 Suppl. P. 361–368. <https://zbmath.org/0913.35050>
44. Samko S.G. Convolution type operators in $L^{p(x)}$ // Integral Transforms and Special Functions. 1998. Vol. 7. Issues 1–2. P. 123–144. <https://doi.org/10.1080/10652469808819191>
45. Шарапудинов И.И. О топологии пространства $\mathcal{L}^{p(t)}([0, 1])$ // Математические заметки. 1979. Т. 26. Вып. 4. С. 613–632. <https://www.mathnet.ru/rus/mzm8442>
46. Шарапудинов И.И. Некоторые вопросы теории приближений в пространствах Лебега с переменным показателем. Владикавказ: Южный математический институт Владикавказского научного центра Российской академии наук и Правительства Республики Северная Осетия–Алания, 2012. <https://www.elibrary.ru/item.asp?id=22887342>
47. Шарапудинов И.И. Приближение функций из пространств Лебега и Соболева с переменным показателем суммами Фурье–Хаара // Математический сборник. 2014. Т. 205. № 2. С. 145–160. <https://doi.org/10.4213/sm8274>
48. Szal B. Trigonometric approximation by Nörlund type means in L^p -norm // Commentationes Mathematicae Universitatis Carolinae. 2009. Vol. 50. Issue 4. P. 575–589. <http://dml.cz/dmlcz/137448>
49. Sonker S., Singh U. Approximation of signals (functions) belonging to $\text{Lip}(\alpha, p, \omega)$ -class using trigonometric polynomials // Procedia Engineering. 2012. Vol. 38. P. 1575–1585. <https://doi.org/10.1016/j.proeng.2012.06.193>

50. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent // *Applicable Analysis*. 2021. Vol. 100. Issue 4. P. 819–834. <https://doi.org/10.1080/00036811.2019.1622680>
51. Testici A. Approximation by Nörlund and Riesz means in weighted Lebesgue spaces with variable exponent // *Communications Faculty Of Science University of Ankara Series A1 Mathematics and Statistics*. 2019. Vol. 68. No. 2. P. 2014–2025. <https://doi.org/10.31801/cfsuasmas.460449>
52. Testici A., Israfilzade D. M. Linear methods of approximation in weighted Lebesgue spaces with variable exponent // *Hacettepe Journal of Mathematics and Statistics*. 2021. Vol. 50. Issue 3. P. 744–753. <https://doi.org/10.15672/hujms.798028>
53. Testici A., Israfilov D. M. Approximation by matrix transforms in Morrey spaces // *Проблемы анализа — Issues of Analysis*. 2021. Т. 10 (28). Вып. 2. С. 79–98. <https://doi.org/10.15393/j3.art.2021.9635>
54. Testici A., Israfilov D. M. Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent // *Journal of Numerical Analysis and Approximation Theory*. 2021. Vol. 50. No. 1. P. 60–72. <https://doi.org/10.33993/jnaat501-1234>
55. Волосивец С. С. Приближение функций и их сопряженных в пространствах Лебега с переменным показателем // *Математический сборник*. 2017. Т. 208. № 1. С. 48–64. <https://doi.org/10.4213/sm8636>
56. Volosivets S. S. Modified modulus of smoothness and approximation in weighted Lorentz spaces by Borel and Euler means // *Проблемы анализа — Issues of Analysis*. 2021. Т. 10 (28). Вып. 1. С. 87–100. <https://doi.org/10.15393/j3.art.2021.8950>
57. Volosivets S. S. Approximation by Vilenkin polynomials in weighted Orlicz spaces // *Analysis Mathematica*. 2021. Vol. 47. Issue 2. P. 437–449. <https://doi.org/10.1007/s10476-021-0086-6>
58. Zygmund A. *Trigonometric series*. Cambridge: Cambridge University Press, 2003. <https://doi.org/10.1017/CBO9781316036587>

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