

Extended higher order operators with the generalized cardinal sine kernel: analytical study and numerical simulations



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Abstract

The extended higher-order fractional derivative operators with the generalized cardinal sine kernel are generalizations of classical higher-order operators, defined in broader function spaces and involving singular kernels. Integration by parts was employed to define these extended derivatives, and the corresponding integral operators were subsequently obtained by solving related fractional differential equations using the Laplace transform. Several properties of the new extended operators are discussed; in particular, it is shown that the extended integral operator serves as the inverse of the extended derivative. Multiple examples are provided to demonstrate the efficiency of the proposed operators. On one hand, it is shown that the extended Caputo-type derivative is well-defined for certain functions where the original derivative fails. On the other hand, solutions to fractional differential equations that are not solvable using traditional derivatives are successfully obtained. A fractional Duffing model involving the extended derivative is considered, and numerical simulations are presented. The proposed extended operators effectively address limitations related to the initialization of fractional differential equations, and the numerical results confirm the efficiency and applicability of the new approach.

Keywords: Caputo derivative, fractional derivatives, fractional differential equations, generalized cardinal sine kernel.

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1. Introduction

Fractional calculus is an emerging and rapidly growing field of mathematics that has been used as a powerful and efficient modeling tool for many applications in many science and engineering fields. The non-locality of fractional derivative operators has made them ideally suitable to describe the long-term memory and hereditary characteristics of many materials and processes in modeling real-world phenomena, which rely on memory effects. Various definitions of fractional derivative operators with different kernels and different application domains have been presented in the literature.

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Basically, in the beginning non-local operators with singular kernels were introduced such as the Riemann-Liouville (RL), the Caputo and the Erdélyi-Kober fractional derivative operators [31, 32, 37, 40]. Later, some non-singular kernel operators such as the Atangana-Baleanu [9], the Caputo-Fabrizio [15], the extended Atangana-Baleanu derivative [6, 32], and the generalized cardinal sine (GCS) kernel [43] fractional derivative operators were presented. The motivation behind defining new fractional derivative models [24, 25, 40, 41] different from those already presented is related to the real data associated with complex systems arising in fractional calculus for which specific kernels are needed. For example, the Atangana-Baleanu (AB) operator was presented based on the use of the Mittag-Leffler function as the kernel of the model [9]. The AB fractional derivative of order α , where $0 < \alpha < 1$, is defined as, for $t > 0$,

$$({}^{\text{AB}}_0 D^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\theta)^\alpha\right) f'(\theta) d\theta,$$

so that B represents a normalization function where $B(0) = B(1) = 1$ and $E_\alpha(z)$ indicates the Mittag-Leffler function which is identified as [3], for $\alpha > 0$ and $z \in \mathbb{C}$,

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha + 1)} z^n.$$

Furthermore, Caputo and Fabrizio (CF) suggested another operator where the exponential function is employed as a non-singular kernel [15]. The CF fractional derivative of order α , where $0 < \alpha < 1$, is described as, for $t > 0$,

$$({}^{\text{CF}}_0 D^\alpha f)(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\theta)\right) f'(\theta) d\theta.$$

In the literature, several studies have addressed the Atangana-Baleanu [17, 35, 39, 42, 53, 59] and Caputo-Fabrizio [1, 8, 12, 19, 26, 46] fractional derivatives as they have been frequently used in many applications of fractional calculus. The theory of generalized fractional operators including singular and non-singular kernels has been discussed in [2, 7, 11, 21, 22, 38]. However, for any non-singular fractional derivative operator of the form $D^\alpha f(t) = \int_0^t k(t, \theta) f'(\theta) d\theta$, there is always data restrictions, namely $\lim_{t \rightarrow 0^+} (D^\alpha f(t)) = 0$ [44]. This drawback complicates the initialization of the relevant fractional differential equations (FDEs) and thus leads to difficulty in providing solutions for these FDEs. In particular, motivated by this limitation, a modified extension of the Atangana-Baleanu operator, which displays a singular kernel, was presented in [6] to address the initialization constraints. The motivation behind using the extended higher-order fractional derivative operators with the generalized cardinal sine kernel lies in addressing specific theoretical and practical limitations found in existing fractional calculus formulations. Traditional fractional derivatives, such as the Caputo or Riemann-Liouville derivatives, often struggle with issues related to singular behavior, limited domain definitions, and difficulties in initialization of fractional differential equations. To overcome these challenges, we proposed an extended operator that incorporates a generalized cardinal sine kernel, allowing for the definition of fractional derivatives in a broader class of functions and improving their applicability to more complex and singular problems. The method also offers a natural way to define the inverse integral operator by leveraging integration by parts and the Laplace transform. This approach ensures consistency between the derivative and integral operators, a crucial property for solving fractional differential equations. Furthermore, our method enables solutions to be obtained for a class of problems (e.g., specific Duffing-type fractional models) that are not solvable using conventional definitions. Introducing and proposing fractional derivative models based on distinct kernels is a key objective for researchers to model the diverse processes and behaviors found in applications of fractional calculus. Recently, in [43], a fractional derivative model was presented relying on a generalized version of the cardinal sine function as the kernel of the model. This model, which is quite different from all the previously introduced models, has the advantage that it can be extended to a singular kernel operator to overcome the limitations of the non-singular version. Fractional

derivatives are used to model several real life models such as blood Alcohol model and dynamics of dengue fever, see [13, 20, 27–30, 36, 47–49, 54, 56, 57] and in fluid dynamics [50–52, 55, 58]. The main aim of the current work is to introduce higher order derivatives of the fractional derivative model with the GCS kernel. The novelty of this paper is to introduce extended higher order fractional derivative and integral operators involving generalized cardinal sine kernels with singular behavior, which significantly broaden the applicability of classical and existing higher-order fractional operators. A key innovation lies in employing integration by parts and Laplace transform techniques to rigorously define these operators and their inverses. The novelty further includes demonstrating that the extended Caputo-type derivative is well-defined for functions where the classical derivative fails. Also, providing solutions to fractional differential equations that are otherwise unsolvable using traditional operators and effectively resolving initialization issues inherent in standard fractional models. In addition, applying the new operators to a fractional Duffing model, with numerical simulations confirming the enhanced stability and efficiency of the proposed framework. Also, these contributions position the new extended operators as powerful tools for both theoretical analysis and practical modeling in fractional calculus.

In Section 2, we present the definitions of the GCS kernel and the extended fractional operators with GCS kernel of order $0 < \alpha < 1$, and some related results. Section 3, is devoted to the definition and main properties of the extended higher order fractional operators with the GCS kernel. We implement integration by parts to expand higher order derivatives with singular kernels to a wider space, and then introduce the extended higher order integral operators. In Section 4, several characteristics of the extended operators and some illustrative examples are presented to demonstrate the advantages of the studied extended operators. Moreover, in order to highlight the benefits of the studied extended operators, we present numerical simulations and demonstrate the dynamics of a certain nonlinear fractional order model involving the studied extended derivatives in Section 5. Finally, Section 6 provides some concluding remarks.

2. The generalized cardinal sine kernel fractional derivative

The generalized cardinal sine kernel is considered in this work due to its distinctive analytical properties that make it well-suited for extending fractional derivative operators, especially in the presence of singularities. Unlike classical kernels, the sinc kernel exhibits both decay and oscillatory behavior, which allows it to regularize singularities and ensure well-defined operators even for functions with limited smoothness. This enables the extension of fractional derivatives to a broader class of functions where traditional definitions, such as the Caputo derivative, may fail. Additionally, the sinc kernel supports the analytical inversion of the derivative operator via the Laplace transform, leading to a consistent formulation of the corresponding integral operator. Its integrability and boundedness also enhance numerical stability, making it advantageous for practical computations. Furthermore, the sinc function's established role in interpolation and signal processing brings intuitive appeal and mathematical rigor to the fractional calculus framework developed in this study. In this section, we present a review of a recently defined fractional derivative operator based on a GCS kernel introduced in [43]. In addition, we provide a review of the associated fractional integral and the extended fractional derivative operators. Some characteristics and relationships of the studied operators are also presented. The description of the considered operators is based on the use of a generalized form of the cardinal sine function, which is defined as

$$\text{sinc}(t) = \begin{cases} \frac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0, \end{cases}$$

as a non-singular kernel. The function g , where $g(t) = \text{sinc}(\frac{\pi}{2}t)$, meets the properties $\int_0^\infty g(t) dt = 1$ and $\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} g(\frac{t}{\epsilon}) = \delta(t)$, where δ indicates the Dirac-delta function. Furthermore, it is an entire function that can be expressed by the power series

$$g(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (t)^{2n}.$$

Now, according to the subsequent generalized form of the cardinal sine function

$$g^\alpha(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\Gamma(2n\alpha+1)} \left(\frac{\pi}{2}t\right)^{2n},$$

the characterization of the GCS kernel fractional derivative is described as follows.

Definition 2.1. The GCS kernel fractional derivative operator of Caputo type ${}_0^C D_g^\alpha$, where $0 < \alpha < 1$, with respect to the kernel g^α is defined as, for $t > 0$,

$$({}_0^C D_g^\alpha)f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t g^\alpha\left(\frac{\alpha}{1-\alpha}(t-\theta)^\alpha\right) f'(\theta) d\theta, \quad (2.1)$$

so that B represents a normalization function, where $B(0) = B(1) = 1$.

In light of the above definition, the operator ${}_0^C D_g^\alpha$ requires the absolute integrability of the function f' , and so Definition 2.1 provides a well-defined fractional derivative operator on the interval $(0, T)$ for all $f \in W^1(0, T)$, where

$$W^1(0, T) = \{f \in C^1[0, T] \mid f' \in L^1(0, T)\}.$$

Next, if we set the function G to be the Taylor series expansion of the ratio $\frac{\tan^{-1}(t)}{t}$,

$$G(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} t^{2n}, \quad (2.2)$$

then, in this case, the GCS kernel fractional derivative operator ${}_0^C D_g^\alpha$ can be characterized as

$$({}_0^C D_g^\alpha)f(t) = \frac{B(\alpha)}{1-\alpha} G\left(\frac{\pi}{2} \frac{\alpha}{1-\alpha} {}_0 I^\alpha\right) \{f(t) - f(0)\} = \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha\right)^{2n} {}_0 I^{2n\alpha} \{f(t) - f(0)\},$$

where ${}_0 I^\beta$ is the RL fractional integral operator of order $\beta > 0$, which is defined as, for $t > 0$,

$$({}_0 I^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f(\theta) d\theta.$$

If we represent the Taylor series expansion of the ratio $\frac{t}{\tan^{-1}(t)}$ by the function $\overline{G}$,

$$\overline{G}(t) = \sum_{n=0}^{\infty} c_n \tau^{2n} = 1 + \frac{1}{3} t^2 - \frac{4}{45} t^4 + \frac{44}{945} t^6 - \frac{428}{14175} t^8 + \dots, \quad (2.3)$$

then we easily recognize that $G(t) \overline{G}(t) = 1$.

Remark 2.2. The power series representation of the function G , introduced in Eq. (2.2), converges on $(0, T)$, for all $T > 0$, and thus the power series representation of $\overline{G}$, introduced in Eq. (2.3), converges on $(0, a)$. Hence, truncated series of (2.2) and (2.3) can be used for approximation purposes.

Definition 2.3. The fractional integral operator ${}_0 I_g^\alpha$, where $0 < \alpha < 1$, related to the GCS kernel fractional derivative operator ${}_0^C D_g^\alpha$, given in Definition 2.1, is defined as, for $t > 0$,

$$({}_0 I_g^\alpha f)(t) = \frac{1-\alpha}{B(\alpha)} \overline{G}\left(\frac{\pi}{2} \frac{\alpha}{1-\alpha} {}_0 I^\alpha\right) f(t) = \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_\alpha\right)^{2n} ({}_0 I^{2n\alpha} f)(t).$$

Indeed, the integral operator, introduced in Definition 2.3, is well-defined on the interval $(0, T)$ for all $f \in L^1(0, T)$. In particular, if $0 < \alpha < 1$ and $f \in W^1(0, T)$ the operators ${}_0^C D_g^\alpha$ and ${}_0 I_g^\alpha$ satisfy the relationship

$$({}_0 I_g^\alpha {}_0^C D_g^\alpha f)(t) = f(t) - f(0).$$

In fact, if $f \in W^1(0, T)$ and $0 < \alpha < 1$, then $({}_0^C D_g^\alpha f)(0) = 0$ and this causes problems associated with the initialization of the relevant FDEs, see [4, 5, 18]. To overcome this limitation, an extended version of the operator ${}_0^C D_g^\alpha$ was introduced in [43] to extend the function space to L^1 . Letting

$$s_\alpha(t, \theta) = \frac{d}{d\theta} (g^\alpha(\gamma_\alpha(t - \theta))) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)\Gamma(2\alpha n)} \left(\frac{\pi}{2}\gamma_\alpha\right)^{2n} (t - \theta)^{2\alpha n - 1},$$

then

$$\frac{\partial s_\alpha}{\partial \theta} = \frac{d^2}{d\theta^2} (g^\alpha(\gamma_\alpha(t - \theta))) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(2n+1)\Gamma(2\alpha n - 1)} \left(\frac{\pi}{2}\gamma_\alpha\right)^{2n} (t - \theta)^{2\alpha n - 2}.$$

Note that $s_\alpha(t, \theta)$ is a singular kernel for $0 < \alpha < \frac{1}{2}$ and $\frac{\partial s_\alpha}{\partial \theta}(t, \theta)$ is a singular kernel for $\frac{1}{2} < \alpha < 1$. Therefore, the extended operator can be obtained by performing integration by parts one time whenever $0 < \alpha < 1/2$ and two times whenever $1/2 < \alpha < 1$, to the integral representation of the GCS kernel fractional derivative given in Eq. (2.1).

Definition 2.4. The extended fractional derivative with the GCS kernel g^α of Caputo type ${}^E C_0 D_g^\alpha$ is defined as

$$({}^E C_0 D_g^\alpha f)(t) = \frac{B(\alpha)}{1 - \alpha} \left(f(t) - g^\alpha(\gamma_\alpha t^\alpha) f(0) - \int_0^t s_\alpha(t, \theta) f(\theta) d\theta \right),$$

when $0 < \alpha < 1/2$, and

$$({}^E C_0 D_g^\alpha f)(t) = \frac{B(\alpha)}{1 - \alpha} \left(f(t) - g^\alpha(\gamma_\alpha t^\alpha) f(0) + \int_0^t \frac{\partial s_\alpha}{\partial \theta}(t, \theta) F(\theta) d\theta \right),$$

when $1/2 < \alpha < 1$, where $\gamma_\alpha = \frac{\alpha}{1-\alpha}$ and $F(t) = \int_0^t f(\theta) d\theta$.

3. Extended higher order fractional operators

In this section we define the extended higher order GCS fractional derivative operators and their associated extended higher order integral operators. We also, establish fundamental results about the new operators.

3.1. Extended higher order fractional derivative

In the following we define the extended higher order fractional derivatives with the GCS kernel. Initially, letting $\delta = \alpha + k - 1$, where $0 < \alpha < 1$, and $k = 2, 3, \dots$, then the higher order fractional derivatives with the GCS kernel can be characterized as

$$({}_0^C D_g^\delta f)(t) = \frac{B(\alpha)}{1 - \alpha} \int_0^t g^\alpha(\gamma_\alpha(t - \theta)^\alpha) f^{(k)}(\theta) d\theta.$$

For $0 < \alpha < \frac{1}{2}$, integration by parts one time of the above equation yields

$$({}^E C_0 D_g^\delta f)(t) = \frac{B(\alpha)}{1 - \alpha} \left(f^{(k-1)}(t) - g^\alpha(\gamma_\alpha t^\alpha) f^{(k-1)}(0) - \int_0^t s_\alpha(t, \theta) f^{(k-1)}(\theta) d\theta \right). \quad (3.1)$$

For $\frac{1}{2} < \alpha < 1$, the kernel $s_\alpha(t, \theta)$ is non-singular and we therefore perform one more integration by parts to the integral in (3.1) to obtain

$$({}^{\text{EC}}_0D_g^\delta f)(t) = \frac{B(\alpha)}{1-\alpha} \left(f^{(k-1)}(t) - g^\alpha(\gamma_\alpha t^\alpha) f^{(k-1)}(0) + s_\alpha(t, 0) f^{(k-2)}(0) + \int_0^t \frac{\partial s_\alpha}{\partial \theta}(t, \theta) f^{(k-2)}(\theta) d\theta \right).$$

Note that the above definition is valid for arbitrary $\delta > 0$, provided that $f^{(-1)}(t) = F(t) = \int_0^t f(s) ds$. We have

$$\begin{aligned} \int_0^t s_\alpha(t, \theta) f^{(k-1)}(\theta) d\theta &= - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} \frac{1}{\Gamma(2\alpha n)} \int_0^t (t-\theta)^{2\alpha n-1} f^{(k-1)}(\theta) d\theta \\ &= - \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} ({}_0I^{2\alpha n} f^{(k-1)})(\theta) \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \int_0^t s_\alpha(t, \theta) f^{(n-2)}(\theta) d\theta &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} \frac{1}{\Gamma(2\alpha n-1)} \int_0^t (t-\theta)^{2\alpha n-2} f^{(k-2)}(\theta) d\theta \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} ({}_0I^{2\alpha n-1} f^{(k-2)})(\theta) \end{aligned} \quad (3.3)$$

and thus we arrive at the formal definition of the extended higher order operators.

Definition 3.1. For $\delta = \alpha + k - 1$, $0 < \alpha < \frac{1}{2}$, and $f^{(k-1)} \in L^1(0, T)$, the extended higher order GCS fractional derivative of Caputo type and order $\delta > 1$, is defined as

$$({}^{\text{EC}}_0D_g^\delta f)(t) = \frac{B(\alpha)}{1-\alpha} \left(f^{(k-1)}(t) - g^\alpha(\gamma_\alpha t^\alpha) f^{(k-1)}(0) + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} ({}_0I^{2\alpha n} f^{(k-1)})(\theta) \right).$$

For $\delta = \alpha + k - 1$, $\frac{1}{2} < \alpha < 1$, and $f^{(k-2)} \in L^1(0, T)$, the extended higher order GCS fractional derivative of Caputo type is defined as

$$\begin{aligned} ({}^{\text{EC}}_0D_g^\delta f)(t) &= \frac{B(\alpha)}{1-\alpha} \left(f^{(k-1)}(t) - g^\alpha(\gamma_\alpha t^\alpha) f^{(k-1)}(0) + s_\alpha(t, 0) f^{(k-2)}(0) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} ({}_0I^{2\alpha n-1} f^{(k-2)})(\theta) \right) \\ &= \frac{B(\alpha)}{1-\alpha} \left(f^{(k-1)}(t) - g^\alpha(\gamma_\alpha t^\alpha) f^{(k-1)}(0) - f^{(k-2)}(0) \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} ({}_0I^{2\alpha n} 1) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\pi}{2} \gamma_\alpha \right)^{2n} ({}_0I_0^{2\alpha n-1} f^{(k-2)})(\theta) \right). \end{aligned}$$

3.2. Extended higher order integral operator

We utilize the Laplace transform to introduce the extended integral operator associated with extended higher order GCS fractional derivative operator ${}^{\text{EC}}_0D_g^\delta f$. Given that

$$({}^{\text{C}}_0D_g^\delta f)(t) = \frac{B(\alpha)}{1-\alpha} g^\alpha(\gamma_\alpha t^\alpha) * f^{(k)}(t),$$

and $(\mathcal{L}g^\alpha(\gamma_\alpha t^\alpha))(s) = \frac{1}{s} G(\frac{\pi}{2}\gamma_\alpha s^{-\alpha})$, we have [57]

$$(\mathcal{L} {}^{\text{EC}}_0 D_g^\delta f)(s) = (\mathcal{L} {}^{\text{C}}_0 D_g^\delta f)(s) = \frac{B(\alpha)}{1-\alpha} \frac{1}{s} G(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}) \left(s^k \hat{f}(s) - \sum_{j=1}^k s^{j-1} f^{(k-j)}(0) \right), \quad (3.4)$$

where $\hat{f}(s) = (\mathcal{L}f)(t)$. To define the inverse integral operator we consider the FDE $({}^{\text{C}}_0 D_g^\delta u)(t) = h(t)$. Performing the Laplace transform to the above FDE yields

$$\frac{B(\alpha)}{1-\alpha} \frac{1}{s} G(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}) \left(s^k \hat{u}(s) - \sum_{j=1}^k s^{j-1} u^{(k-j)}(0) \right) = \hat{h}(s)$$

or

$$s^{k-1} G(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}) \hat{u}(s) = G(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}) \sum_{j=1}^k f^{(k-j)}(0) s^{j-2} + \frac{1-\alpha}{B(\alpha)} \hat{h}(s)$$

or

$$\hat{u}(s) = \sum_{j=1}^k f^{(k-j)}(0) s^{j-k-1} + \frac{1-\alpha}{B(\alpha)} \frac{1}{s^{k-1}} \bar{G}(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}) \hat{h}(s),$$

and thus

$$\begin{aligned} u(t) &= \sum_{j=1}^k f^{(k-j)}(0) \frac{t^{k-j}}{\Gamma(k-j+1)} + \frac{1-\alpha}{B(\alpha)} {}_0 I^{k-1} \mathcal{L}^{-1} \left(\bar{G}(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}) \hat{h}(s) \right) \\ &= \sum_{j=1}^k f^{(k-j)}(0) \frac{t^{k-j}}{\Gamma(k-j+1)} + \frac{1-\alpha}{B(\alpha)} {}_0 I^{k-1} \left(\sum_{n=0}^{\infty} c_n (\frac{\pi}{2}\gamma_\alpha)^{2n} {}_0 I^{2\alpha n} h \right)(t) \\ &= \sum_{j=1}^k f^{(k-j)}(0) \frac{t^{k-j}}{(k-j)!} + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n (\frac{\pi}{2}\gamma_\alpha)^{2n} ({}_0 I^{2\alpha n+k-1} h)(t). \end{aligned}$$

From the last equation we have for $\delta = \alpha + n - 1$,

$$({}_0 I_g^\delta h)(t) = \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n (\frac{\pi}{2}\gamma_\alpha)^{2n} ({}_0 I^{2\alpha n+k-1} h)(t).$$

Definition 3.2. For $\delta = \alpha + k - 1$, $0 < \alpha < 1$, $k \geq 1$, and $f \in L^1(0, T)$, the integral operator ${}_0 I_g^\delta$ is defined as

$$({}_0 I_g^\delta f)(t) = \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n (\frac{\pi}{2}\gamma_\alpha)^{2n} ({}_0 I^{2\alpha n+k-1} f)(t).$$

3.3. Fundamental result on extended higher order fractional operators

We introduce the relationship between the higher order version of the extended integral operator ${}_0 I_g^\delta$ and the extended GCS fractional derivative operator ${}^{\text{EC}}_0 D_g^\delta f$ with respect to the concept of inversion between the two operators. The Laplace transform of the fractional integral operator ${}_0 I_g^\delta$ is given by

$$\begin{aligned} \mathcal{L}({}_0 I_g^\delta h)(s) &= \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n (\frac{\pi}{2}\gamma_\alpha)^{2n} \mathcal{L} \left({}_0 I^{2\alpha n+k-1} h \right)(s) \\ &= \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n (\frac{\pi}{2}\gamma_\alpha)^{2n} \frac{1}{s^{2\alpha n+k-1}} \mathcal{L}(h)(s) \end{aligned}$$

$$= \frac{1-\alpha}{B(\alpha)} \frac{1}{s^{k-1}} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right)^{2n} \mathcal{L}(h)(s) = \frac{1-\alpha}{B(\alpha)} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \frac{(\mathcal{L}h)(s)}{s^{k-1}}.$$

In the upcoming result, we demonstrate that the extended fractional integral operator ${}_0I_g^{\delta}$ is an inverse operator of the extended GCS fractional derivative operator ${}^{\text{EC}}{}_0D_g^{\delta}f$.

Theorem 3.3. Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $\delta = \alpha + k - 1$, where $0 < \alpha < 1$.

(1) If ${}^{\text{EC}}{}_0D_g^{\delta}f \in L^1(0, T)$, then

$$({}_0I_g^{\delta} {}^{\text{EC}}{}_0D_g^{\delta}f)(t) = f(t) - \sum_{j=1}^k \frac{f^{(k-j)}(0)}{(k-j)!} t^{k-j}.$$

(2) If $f^{(k-1)} \in L^1(0, T)$, then

$$({}^{\text{EC}}{}_0D_g^{\delta} {}_0I_g^{\delta}f)(t) = f(t) - g^{\alpha}(\gamma_{\alpha} t^{\alpha})f(0).$$

Proof.

(1) We have

$$\begin{aligned} \mathcal{L}({}_0I_g^{\delta} {}^{\text{EC}}{}_0D_g^{\delta}f)(s) &= \frac{1-\alpha}{B(\alpha)} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \frac{1}{s^{k-1}} \mathcal{L}({}^{\text{EC}}{}_0D_g^{\delta}f)(s), \\ &= \frac{1-\alpha}{B(\alpha)} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \frac{1}{s^{k-1}} \frac{B(\alpha)}{1-\alpha} \frac{1}{s} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \mathcal{L}(f^{(k)})(s), \\ &= \frac{1}{s^k} \mathcal{L}(f^{(k)})(s) = \mathcal{L}(f)(s) - \sum_{j=1}^k \frac{f^{(k-j)}(0)}{s^{k-j+1}}. \end{aligned}$$

Applying the inverse Laplace transform to the above equation yields

$$({}_0I_g^{\delta} {}^{\text{EC}}{}_0D_g^{\delta}f)(t) = f(t) - \sum_{j=1}^k \frac{f^{(k-j)}(0)}{(k-j)!} t^{k-j},$$

which completes the proof of (1).

(2) We have

$$\begin{aligned} \mathcal{L}({}^{\text{EC}}{}_0D_g^{\delta} {}_0I_g^{\delta}f)(s) &= \frac{B(\alpha)}{1-\alpha} \frac{1}{s} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \left(s^k \mathcal{L}({}_0I_g^{\delta}f)(s) - \sum_{j=1}^k s^{j-1} ({}_0I_g^{\delta}f)^{(k-j)}(0) \right) \\ &= \frac{B(\alpha)}{1-\alpha} \frac{1}{s} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \left(s^k \frac{1-\alpha}{B(\alpha)} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \frac{(\mathcal{L}f)(s)}{s^{k-1}} - \sum_{j=1}^k s^{j-1} ({}_0I_g^{\delta}f)^{(k-j)}(0) \right) \quad (3.5) \\ &= (\mathcal{L}f)(s) - \frac{B(\alpha)}{1-\alpha} \frac{1}{s} \overline{G} \left(\frac{\pi}{2} \gamma_{\alpha} s^{-\alpha} \right) \sum_{j=1}^k s^{j-1} ({}_0I_g^{\delta}f)^{(k-j)}(0). \end{aligned}$$

Because $f' \in L^1(0, T)$, then $({}_0I^{\beta}f)(0) = 0$, $\beta > 0$, which together with $({}_0I^0f)(0) = f(0)$, will lead to

$$D^{k-j}({}_0I_g^{\delta}f)(t) = D^{k-j} \left(\frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_{\alpha} \right)^{2n} ({}_0I^{2\alpha n+k-1}f)(t) \right) = \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_{\alpha} \right)^{2n} ({}_0I^{2\alpha n+j-1}f)(t).$$

We have

$$\sum_{j=1}^k s^{j-1} ({}_0I_g^\delta f)^{(k-j)}(0) = \frac{1-\alpha}{B(\alpha)} c_0 f(0) = \frac{1-\alpha}{B(\alpha)} f(0).$$

Substituting the above result in Eq. (3.5) yields

$$\mathcal{L}({}^{\text{EC}}{}_0D_g^\delta {}_0I_g^\delta f)(s) = (\mathcal{L}f)(s) - \frac{B(\alpha)}{1-\alpha} \frac{1}{s} G\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) \frac{1-\alpha}{B(\alpha)} f(0) = (\mathcal{L}f)(s) - \frac{1}{s} G\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) f(0).$$

Applying the inverse Laplace operator to the above equations we arrive at

$$({}^{\text{EC}}{}_0D_g^\delta {}_0I_g^\delta f)(t) = f(t) - g^\alpha(\gamma_\alpha t^\alpha) f(0),$$

which completes the proof of (2). $\square$

4. Motivation to the extension and related FDEs

In this section, we provide several illustrative examples to demonstrate the efficiency of the extended derivatives. We also, present in closed forms the solutions of related FDEs with the extended derivatives.

Example 4.1. Consider

$$f(t) = \begin{cases} t^{\frac{3}{2}} \sin(t^{-\frac{1}{3}}), & t > 0, \\ 0, & t = 0. \end{cases}$$

We have $f \in C[0, T]$ and $f'(t) = \frac{3}{2} t^{\frac{1}{2}} \sin(t^{-\frac{1}{3}}) - \frac{1}{3} t^{\frac{1}{6}} \cos(t^{-\frac{1}{3}})$, $t > 0$, is integrable on $[0, T]$, with $f'(0) = 0$. So, ${}^{\text{EC}}{}_0D_g^\delta f$ is well defined for any $1 < \delta < 2$. Direct calculation will lead to the fact that $f''(t)$ is not integrable on $[0, t]$, for arbitrary $t > 0$. That is, ${}_0D_g^\delta f$ is not well-defined for $\delta > 1$.

Example 4.2. For $1 < \delta < 2$, the solution of the IVP

$$({}^{\text{EC}}{}_0D_g^\delta u)(t) = 1, \quad t > 0, \quad u(0) = u_0, \quad u'(0) = -\frac{1-\alpha}{B(\alpha)}, \quad (4.1)$$

is given by

$$u(t) = u_0 - \frac{1-\alpha}{B(\alpha)} t + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_\alpha\right)^{2n} \frac{t^{2\alpha n+1}}{\Gamma(2\alpha n+2)}. \quad (4.2)$$

Proof. One can easily verify that the solution presented in Eq. (4.2) satisfies the initial conditions in Eq. (3.2). We have

$$(\mathcal{L}u)(s) = \frac{u_0}{s} - \frac{1-\alpha}{B(\alpha)} \frac{1}{s^2} + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_\alpha\right)^{2n} \frac{1}{s^{2\alpha n+2}} = \frac{u_0}{s} - \frac{1-\alpha}{B(\alpha)} \frac{1}{s^2} + \frac{1-\alpha}{B(\alpha)} \frac{1}{s^2} \overline{G}\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right),$$

and thus

$$s^2(\mathcal{L}u)(s) = u_0 s - \frac{1-\alpha}{B(\alpha)} + \frac{1-\alpha}{B(\alpha)} \overline{G}\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right).$$

Now, using the result in (3.4) we have

$$\begin{aligned} (\mathcal{L} {}^{\text{EC}}{}_0D_g^\delta u)(s) &= \frac{B(\alpha)}{1-\alpha} \frac{1}{s} G\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) (s^2(\mathcal{L}u)(s) - su'(0) - u(0)), \\ &= \frac{B(\alpha)}{1-\alpha} \frac{1}{s} G\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) \frac{1-\alpha}{B(\alpha)} \overline{G}\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) = \frac{1}{s} = (\mathcal{L}1)(s), \end{aligned}$$

which completes the proof. $\square$

Remark 4.3. In Eq. (3.2) if we replace ${}^{\text{EC}}_0D_g^\delta$ by ${}_C_0D_g^\delta$, then the problem has no solution. Because $({}_C_0D_g^\delta u)(0) = 0$, a necessary condition to have a solution to Eq. (3.2) is that, the terms on the right hand side of the equation has to vanish at $t = 0$, see [4, 5]. Also, the obtained solution $u(t)$ in Eq. (4.2) satisfies $u' \in L^1[0, T]$, while $u'' \notin L^1[0, T]$, and therefore $({}_C_0D_g^\delta u)(t)$ is not well-defined.

The solution of Eq. (3.2) can be obtained for arbitrary $\delta > 1$, as follows.

Example 4.4. For $k - 1 < \delta < k$, $k \in \mathbb{N}$, the solution of the IVP

$$({}^{\text{EC}}_0D_g^\delta u)(t) = 1, \quad t > 0, \quad u(0) = r_0, \quad u'(0) = r_1, \dots, \quad u^{(k-2)}(0) = r_{k-2}, \quad u^{(k-1)}(0) = -\frac{1-\alpha}{B(\alpha)}, \quad (4.3)$$

is given by

$$u(t) = r_0 + \frac{r_1}{1!}t + \frac{r_2}{2!}t^2 + \dots + \frac{r_{k-2}}{(k-2)!}t^{k-2} - \frac{1-\alpha}{B(\alpha)} \frac{t^{k-1}}{(k-1)!} + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2}\gamma_\alpha\right)^{2n} \frac{t^{2\alpha n+k-1}}{\Gamma(2\alpha n+k)}. \quad (4.4)$$

Proof. One can verify that the solution presented in (4.4) satisfies the initial conditions in (4.3). Also,

$$\begin{aligned} (\mathcal{L}u)(s) &= \frac{r_0}{s} + \frac{r_1}{s^2} + \dots + \frac{r_{k-2}}{s^{k-1}} - \frac{1-\alpha}{B(\alpha)} \frac{1}{s^k} + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2}\gamma_\alpha\right)^{2n} \frac{1}{s^{2\alpha n+k}}, \\ &= \frac{r_0}{s} + \frac{r_1}{s^2} + \dots + \frac{r_{k-2}}{s^{k-1}} - \frac{1-\alpha}{B(\alpha)} \frac{1}{s^k} + \frac{1-\alpha}{B(\alpha)} \frac{1}{s^k} \bar{G}\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right), \end{aligned}$$

and thus

$$s^k(\mathcal{L}u)(s) - \sum_{j=1}^k s^{j-1}u^{(k-j)}(0) = s^k(\mathcal{L}u)(s) - r_0s^{k-1} - r_1s^{k-2} - \dots - r_{k-2}s + \frac{1-\alpha}{B(\alpha)}, = \frac{1-\alpha}{B(\alpha)} \bar{G}\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right).$$

Substituting the above result in Eq. (3.4) will lead to

$$(\mathcal{L} {}^{\text{EC}}_0D_g^\delta u)(s) = \frac{B(\alpha)}{1-\alpha} \frac{1}{s} G\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right) \frac{1-\alpha}{B(\alpha)} \bar{G}\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right) = \frac{1}{s},$$

and the result follows by performing the inverse Laplace transform. $\square$

Lemma 4.5. For $1 < \delta < 2$, the solution of the sequential fractional differential equation

$$({}^{\text{EC}}_0D_g^\delta u)(t) + ({}^{\text{EC}}_0D_g^{\delta-1}u)(t) = h(t), \quad (4.5)$$

is given by

$$u(t) = u(0) + u'(0)(1 - e^{-t}) + e^{-t} * ({}_0I_g^\alpha h)(t),$$

provided that $\hat{h}(s) = (\mathcal{L}h)(s)$ is well-defined.

Proof. Applying the Laplace transform to Eq. (4.5) we have

$$\frac{B(\alpha)}{1-\alpha} \frac{1}{s} G\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right) (s^2 \hat{u}(s) - s u(0) - u'(0)) + \frac{B(\alpha)}{1-\alpha} \frac{1}{s} G\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right) (s \hat{u}(s) - u(0)) = \hat{h}(s),$$

or

$$s^2 \hat{u}(s) - s u(0) - u'(0) + s \hat{u}(s) - u(0) = \frac{1-\alpha}{B(\alpha)} \frac{s}{G\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right)} \hat{h}(s) = \frac{1-\alpha}{B(\alpha)} s \bar{G}\left(\frac{\pi}{2}\gamma_\alpha s^{-\alpha}\right) \hat{h}(s).$$

Direct calculations lead to

$$\hat{u}(s) = \frac{u(0)}{s} + \frac{u'(0)}{s(s+1)} + \frac{1}{s+1} \frac{1-\alpha}{B(\alpha)} \bar{G}\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) \hat{h}(s).$$

Because

$$\frac{1-\alpha}{B(\alpha)} \bar{G}\left(\frac{\pi}{2} \gamma_\alpha s^{-\alpha}\right) \hat{h}(s) = \mathcal{L}({}_0 I_g^\delta u)(s),$$

we have

$$\hat{u}(s) = \frac{u(0)}{s} + u'(0) \left(\frac{1}{s} - \frac{1}{s+1}\right) + \frac{1}{s+1} \mathcal{L}({}_0 I_g^\delta u)(s).$$

The result follows directly by performing the inverse Laplace operator to the above equation. $\square$

5. A nonlinear fractional model

In this section, we describe the dynamic behavior of a nonlinear fractional order model incorporating the studied extended fractional derivative of order $1 < \delta < 2$. We provide approximate numerical solutions for the fractional Duffing equation

$${}^{\text{EC}}_0 D_g^\delta u(t) + au(t) + bu^3(t) = 0, \quad 0 < t < T, \quad (5.1)$$

with the initial conditions

$$u(0) = 1, \quad u'(0) = A, \quad (5.2)$$

where ${}^{\text{EC}}_0 D_g^\delta$ is the extended higher order fractional derivative operator with GCS kernel of order δ , such that $1 < \delta < 2$, $T > 0$, and $a, b, A \in \mathbb{R}$. Referring to first rule given in Theorem 3.3, the fractional Duffing Eq. (5.1) together with the initial conditions given in Eq. (5.2) is equivalent to the integral equation

$$u(t) = 1 + At + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_\alpha\right)^{2n} {}_0 I^{2n\alpha+1}(-au - bu^3)(t), \quad (5.3)$$

where $\alpha = \delta - 1$. Applying the first order ordinary derivative operator $\frac{d}{dt}$ to Eq. (5.3), we obtain

$$u'(t) = A + \frac{1-\alpha}{B(\alpha)} \sum_{n=0}^{\infty} c_n \left(\frac{\pi}{2} \gamma_\alpha\right)^{2n} {}_0 I^{2n\alpha}(-au - bu^3)(t).$$

Now, we will numerically simulate the solution $u(t)$ and $u'(t)$ using a uniform grid on the interval $[0, T]$. Our numerical scheme defines the grid of points $\{t_j : j = 0, 1, \dots, N\}$, where $t_j = jh$ and $h = \frac{T}{N}$ such that $N \in \mathbb{N}$ over the interval $[0, T]$. In [43], a numerical algorithm was proposed for the numerical solution of FDEs involving extended fractional derivatives with GCS kernel of order $0 < \alpha < 1$. By following the basic steps of this algorithm, using the truncated series $\sum_{n=0}^M (\cdot)$, where $M \in \mathbb{N}$, as an approximation to infinite series representation of the function $\bar{G}$, given in Eq. (2.3), we can obtain the numerical approximations $\mu_{k+1} \approx u(t_{k+1})$ and $v_{k+1} \approx u'(t_{k+1})$, for $k = 1, 2, \dots, N-1$, as

$$\begin{aligned} \mu_{k+1} &= 1 + A t_{k+1} + \frac{1-\alpha}{M(\alpha)} \left\{ \sum_{n=0}^M c_n \left(\frac{\pi \gamma_\alpha}{2}\right)^{2n} \left[\frac{h^{2n\alpha+1}}{\Gamma(2n\alpha+3)} (-a \mu_{k+1}^p - b (\mu_{k+1}^p)^3) \right. \right. \\ &\quad \left. \left. + \frac{h^{2n\alpha+1}}{\Gamma(2n\alpha+3)} \sum_{j=0}^k \rho_{n,j,k+1} (-a \mu_j - b (\mu_j)^3) \right] \right\}, \\ v_{k+1} &= A + \frac{1-\alpha}{M(\alpha)} \left\{ \sum_{n=0}^M c_n \left(\frac{\pi \gamma_\alpha}{2}\right)^{2n} \left[\frac{h^{2n\alpha}}{\Gamma(2n\alpha+2)} (-a v_{k+1}^p - b (v_{k+1}^p)^3) \right. \right. \end{aligned}$$

$$+ \frac{h^{2n\alpha}}{\Gamma(2n\alpha+2)} \sum_{j=0}^k \sigma_{n,j,k+1} \left(-a v_j - b (v_j)^3 \right) \Big] \Big\},$$

where $\mu_0 = 1$, $v_0 = A$,

$$\begin{aligned} \mu_{k+1}^p &= 1 + A t_{k+1} + \frac{1-\alpha}{M(\alpha)} \left\{ \sum_{n=0}^M c_n \left(\frac{\pi \gamma_\alpha}{2} \right)^{2n} \frac{h^{2n\alpha+1}}{\Gamma(2n\alpha+2)} \right. \\ &\quad \times \sum_{j=0}^k [(k+1-j)^{2n\alpha+1} - (k-j)^{2n\alpha+1}] (-a \mu_j - b (\mu_j)^3) \Big\}, \\ v_{k+1}^p &= A + \frac{1-\alpha}{M(\alpha)} \left\{ \sum_{n=0}^M c_n \left(\frac{\pi \gamma_\alpha}{2} \right)^{2n} \frac{h^{2n\alpha}}{\Gamma(2n\alpha+1)} \sum_{j=0}^k [(k+1-j)^{2n\alpha} - (k-j)^{2n\alpha}] (-a v_j - b (v_j)^3) \right\}, \\ \rho_{n,j,k+1} &= \begin{cases} k^{2n\alpha+2} - (k-2n\alpha-1)(k+1)^{2n\alpha+1}, & \text{if } j=0, \\ (k-j+2)^{2n\alpha+2} + (k-j)^{2n\alpha+2} - 2(k-j+1)^{2n\alpha+2}, & \text{if } 1 \leq j \leq k, \end{cases} \\ \sigma_{n,j,k+1} &= \begin{cases} k^{2n\alpha+1} - (k-2n\alpha)(k+1)^{2n\alpha}, & \text{if } j=0, \\ (k-j+2)^{2n\alpha+1} + (k-j)^{2n\alpha+1} - 2(k-j+1)^{2n\alpha+1}, & \text{if } 1 \leq j \leq k. \end{cases} \end{aligned}$$

Now, we perform numerical simulations to display the dynamic behavior of problem (5.1)-(5.2). Figures 1 and 2 display numerical approximations of the solution $u(t)$ and $u'(t)$ of the considered model, where $A = 1/2$ when $a = -2$ and $b = 1$. Figures 3 and 4 display numerical approximations of the solution $u(t)$ and $u'(t)$ of the considered model, where $A = 0$ when $a = -2$ and $b = 1$. The numerical approximations are obtained where $\delta = 1.4$ and $\delta = 1.8$ when $T = 1.5$, $h = 0.004$ and $M = 20$.

From the plots shown in Figure 2, we can observe that $u'(t)$ is not continuous at $t = 0$, and so the obtained solution satisfies $u'(t) \in L^1[0, T]$. In this case, when $A = 1/2$, if the extended operator ${}^{\text{EC}}D_g^\delta$ is replaced by the operator ${}^{\text{C}}D^\delta$ in problem (5.1)-(5.2), then the resulting problem has no solution on the space $W^1[0, T]$. From the plots shown in Figure 4, we can observe that $u'(t)$ is continuous at $t = 0$, and so the obtained solution satisfies $u'(t) \in W^1[0, T]$. In this case, when $A = 0$, if the extended operator ${}^{\text{EC}}D^\delta$ is replaced by the operator ${}^{\text{C}}D^\delta$ in problem (5.1)-(5.2), then the resulting problem has the same solution as problem (5.1)-(5.2).

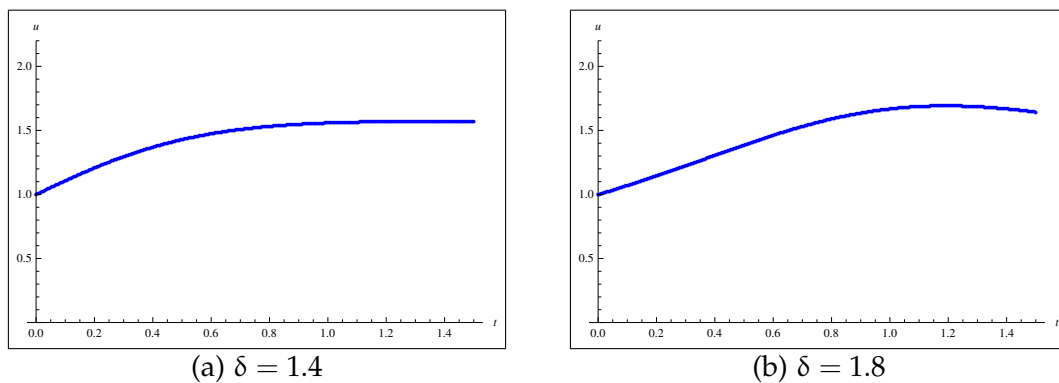


Figure 1: Plots of numerical approximations of the solution $u(t)$ to problem (5.1)-(5.2) with $A = 0.5$, where $a = -2$ and $b = 1$ when $h = 0.005$, $M = 20$ and $T = 1.5$.

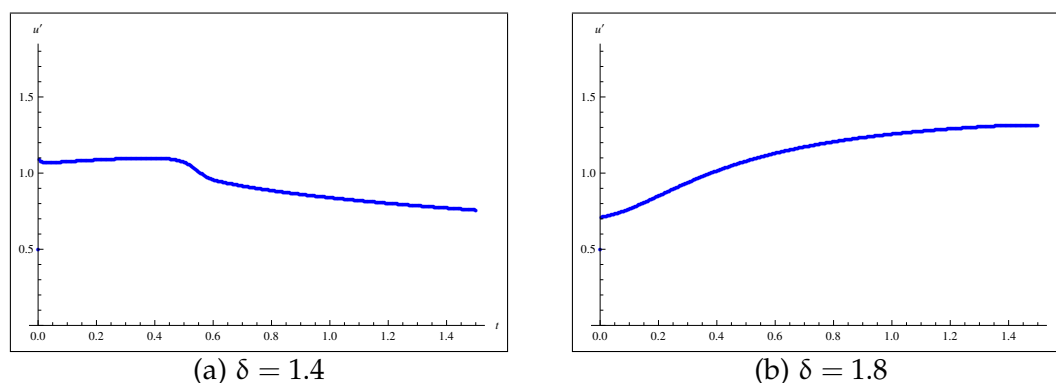


Figure 2: Plots of numerical approximations of $u'(t)$ to problem (5.1)-(5.2) with $A = 0.5$, where $a = -2$ and $b = 1$ when $h = 0.005$, $M = 20$ and $T = 1.5$.

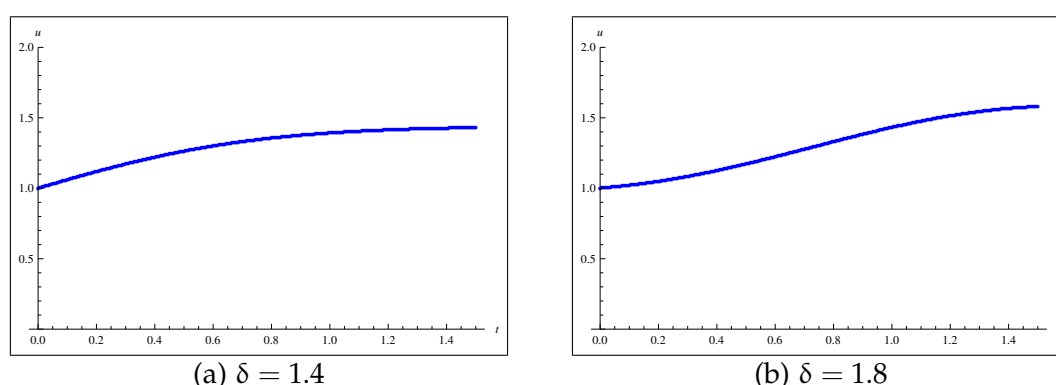


Figure 3: Plots of numerical approximations of the solution $u(t)$ to problem (5.1)-(5.2) with $A = 0$, where $a = -2$ and $b = 1$ when $h = 0.005$, $M = 20$ and $T = 1.5$.

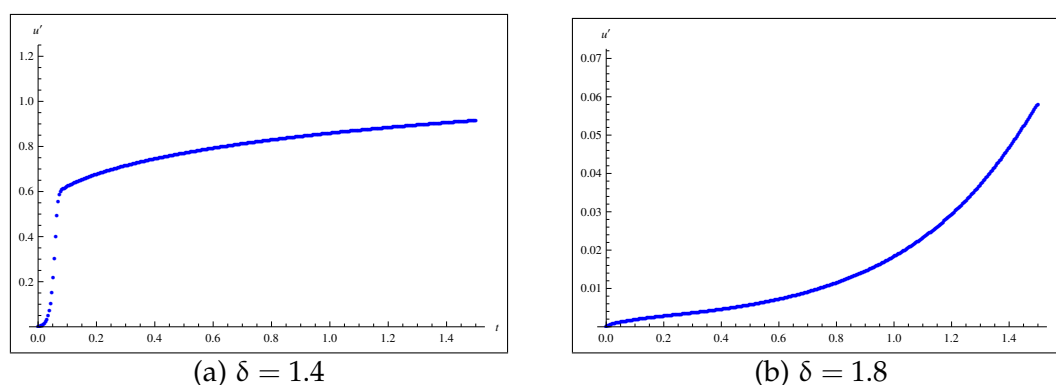


Figure 4: Plots of numerical approximations of $u'(t)$ to problem (5.1)-(5.2) with $A = 0$, where $a = -2$ and $b = 1$ when $h = 0.005$, $M = 20$ and $T = 1.5$.

6. Concluding remarks

This study is devoted to the definition, properties, motivation and applications of extended higher order fractional operators with the GCS kernel. As the GCS kernel is a non-singular kernel, and due to the limitation in solving FDEs with fractional derivative involving non-singular kernels, an extension of higher order fractional operators with GCS kernel is proposed. The new extended operators are defined in wider spaces in contrast to the original ones. The efficiency of the new extended operators is tested

through different aspects, in one side we show that certain FDEs are solvable with the extended derivatives while the same equations admit no solutions with the original derivatives, and on other side the new extended operators can be implemented in modeling several dynamical systems. The analytical and numerical results of the fractional Duffing equation with the extended derivatives illustrate this aspect. We also provide, in closed forms, the solutions of related FDEs and sequential FDEs. Evolution equations are essential in modeling various phenomena across in engineering and natural sciences. The methods and operators proposed in this article have possible applications in modeling dynamical systems; see, e.g., [10, 14, 16, 23, 33, 34]. Future research may focus on extending the proposed higher order fractional operators with generalized cardinal sine kernels to multidimensional domains and partial differential equations, enabling broader applications in physics and engineering. Analytical and approximate solutions for a wider class of nonlinear or coupled fractional differential equations could also be explored. A comparative analysis with other generalized fractional operators, such as the Atangana-Baleanu or Prabhakar derivatives, would offer insights into the advantages and limitations of the proposed approach. Further investigation into the spectral and functional properties of the operators, including their boundedness and eigenstructure, would enhance their theoretical foundation. On the computational side, the development of efficient numerical schemes specifically adapted to the singular behavior of the new operators is essential. Practical applications in viscoelastic materials, anomalous transport, and biological systems should be pursued to validate the model's physical relevance. Moreover, integrating the new operators into optimal control formulations and machine learning frameworks, such as physics-informed neural networks, could open new avenues in data-driven modeling and control of complex fractional systems.

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