

Multimetric Spaces

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ABSTRACT

In the context of multisets, concepts such as multinumber, multimetric, multinorm, and multitopology have been examined, leading to the development of analytical tools that involve inequalities with a finite number of strengthened layers of truth. By establishing certain connections between these structures, the concepts have been clarified, thus laying the foundation for a theory open to further development.

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1 Introduction

Multisets are mathematical entities that both generalize and enrich sets, which constitute one of the most fundamental concepts in mathematics. An element may or may not belong to a set, but when it does, the question of how many instances of that element are present lies beyond the scope of classical set theory. Multisets, however, not only provide information about the elements but also indicate how many of each are present. This characteristic provides a more comprehensive tool for some mathematical models, making multisets a key concept, particularly in discrete mathematics.

Multisets frequently appear in combinatorics in contexts where elements may occur repeatedly, such as in permutations and combinations, and they offer advanced counting techniques to solve complex problems

[8, 37, 38]. In connection with this, multisets also have a broad range of applications in probability theory. In the context of storing and processing repeated data, multisets are useful in nearly all branches of statistics and also in computer science in many subjects such as decision making [30, 31, 46], automata theory [12, 16], coding theory [35], formal methods [34], artificial intelligence [36], multi-agent systems [7], formal languages [12], and they find applications in many other diverse areas [4, 9, 10].

On the other hand, multisets are studied mathematically in their own right and are explored in algebraic [2, 3, 5, 19, 28, 43, 49, 53], logical and set theoretic [6, 11, 18, 29, 41, 48], graph theoretic [9, 10, 22], topological [21, 27, 50], lattice theoretic [30, 55], and analytical [15] contexts. Multisets can be thought of as philosophically, if not formally, dual to fuzzy sets, considering that one strengthens the degrees of membership while the other weakens it. Ultimately, there is a fuzzy multiset concept that combines both approaches [39]. However, research on multiset-based mathematical concepts remains limited, compared to the advancement and proliferation of fuzzy mathematics.

Here, we introduce multinumbers, which are finite multisets of real numbers, independently of other concepts with similar names in the literature, and we construct a partial ordering and algebraic operations such as addition, and scalar multiplication on them. By extending real numbers in this way, we gain the ability to assign multilayered logical truth values to inequalities. This approach gives rise to the concept of multimetric, which is a tool that allows us to assign more robust or reliable layers of truth, in the sense that while it is true that the distance between two points is smaller than a certain value, a higher degree of truth is yet possible for being less than a larger value.

In addition to multimetric spaces, we describe multinormed spaces. In this way, we introduce structures that make sense combinatorially within the context of functional analysis and topology [23, 24, 32, 33, 42]. In particular, we explore multimetric spaces from three different topological perspectives using some special types of sets and multisets. Namely, we observe that multimetrics generate a topology in the classical sense, they generate a topology-like structure on multisets that is quite similar to those found in the literature [21], and they produce a layered topology-like structure on sets, which we call multitopology. The first approach allows multimetric spaces to be examined through the lens of topology, the second approach attributes topological properties to multisets, and the third approach strengthens classical topological concepts by assigning a multiplicity to them. In addition, we examine the concept of convergence in related structures, whose generalizations can also be discussed in the future, and connect it with various concepts of closure.

2 Preliminaries

In this section, the preferred notations related to multisets and strings have been specified. For a more detailed discussion on multisets, one can refer to [39, 54]. The content of this section has been compiled from [25, 54].

One can intuitively think multisets as generalized sets, in which elements can possibly have more than one instances. More formally; a **multiset**, or a bag, over a fixed set X (the universe), is a function $\mathcal{A} : X \rightarrow \mathbb{N}$. We call $\mathcal{A}(x)$ the number of instances of x in $\mathcal{A}$, the degree of membership of x to $\mathcal{A}$, or the multiplicity of x in $\mathcal{A}$. If $\mathcal{A}(x) = 0$, then it is written $x \notin \mathcal{A}$. If $k \in \mathbb{Z}^+$ and $\mathcal{A}(x) \geq k$, we write $x \in^k \mathcal{A}$, which reads as $\mathcal{A}$ has at least k instances of x , and in particular we simply write $x \in \mathcal{A}$ for $x \in^1 \mathcal{A}$. Hence if $k, \ell \in \mathbb{Z}^+$ and $k \geq \ell$, then $x \in^k \mathcal{A} \implies x \in^\ell \mathcal{A}$.

Just like with sets, we write multisets in list form, but using brackets [and] instead of braces to avoid confusion with ordinary sets, and each element appears in this list exactly as many times as the number of its instances in the multiset, while their ordering is disregarded. That is, we have $[a, a, b] = [a, b, a]$, but $[a, a, b] \neq [a, b, b]$. When the context seems to be clear, we occasionally define multisets by properties.

In this context, for instance, the multiset $[x \in \mathbb{Z} : x^6 - x^5 - x^4 + x^3 = 0]$ is equal to $[0, 0, 0, 1, 1, -1]$, when the multiplicities of roots of the polynomial are identified to the number of occurrences of elements in the multiset. However, this practical style should be used carefully and with clear context, as it lacks a formal foundation.

Each ordinary subset A of X can be regarded as a multiset over X , where

$$A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

for all $x \in X$. In particular, the multiset corresponding to the empty set $\emptyset$ is called the **empty multiset**, and we denote it again by $\emptyset$. While we denote the family of all subsets of X by $\mathcal{P}(X)$, the family of all multisets over X is denoted by $\mathcal{M}(X)$. We also use the notation $\mathcal{A} \in X$ to express that $\mathcal{A}$ is a multiset over X . That is,

$$\mathcal{A} \in X : \iff \mathcal{A} \in \mathcal{M}(X).$$

Given two multisets $\mathcal{A}, \mathcal{B} \in X$. The operations intersection, union, addition, and subtraction are defined as follows:

$$\begin{aligned} (\mathcal{A} \cap \mathcal{B})(x) &= \min(\mathcal{A}(x), \mathcal{B}(x)) \\ (\mathcal{A} \cup \mathcal{B})(x) &= \max(\mathcal{A}(x), \mathcal{B}(x)) \\ (\mathcal{A} \oplus \mathcal{B})(x) &= \mathcal{A}(x) + \mathcal{B}(x) \\ (\mathcal{A} \ominus \mathcal{B})(x) &= \max(\mathcal{A}(x) - \mathcal{B}(x), 0) \end{aligned}$$

Moreover, arbitrary intersection of a nonempty family $\{\mathcal{A}_i : i \in I\}$ is given by

$$\left(\bigcap_{i \in I} \mathcal{A}_i \right)(x) = \min_{i \in I} \mathcal{A}_i(x)$$

On the other hand, since a multiset can only contain a finite number of instances of an element, arbitrary unions and sums of multisets are not always defined. However, if I is an index set and both $\mathcal{A}$ and each $\mathcal{A}_i$ for $i \in I$ are multisets over a universe X , then the following notations are used: If for every $x \in X$, the equality $\mathcal{A}(x) = \max_{i \in I} \mathcal{A}_i(x)$ is satisfied, then we write $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$. If for every $x \in X$, the equality $\mathcal{A}(x) = \sum_{i \in I} \mathcal{A}_i(x)$ is satisfied, then we write $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$, where the sum is defined as $\mathcal{A}$ is a multiset, so that each $x \in X$ must belong only to finitely many $\mathcal{A}_i$'s.

For two multisets $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \subset \mathcal{B}$ if $\mathcal{A}(x) \leq \mathcal{B}(x)$ for all $x \in X$, and in this case $\mathcal{A}$ is called a **submultiset** of $\mathcal{B}$. Note that

$$\mathcal{A} \subset \mathcal{B} \iff \forall k \in \mathbb{Z}^+ (x \in^k \mathcal{A} \implies x \in^k \mathcal{B}).$$

We consistently prefer the symbol $\subset$ for multisets, and the symbol $\subseteq$ for sets, throughout the text.

To fix the notations, for instance, let $X = \{a, b, c\}$, and define the multisets $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}(a) = \mathcal{A}(b) = 2$, $\mathcal{A}(c) = 0$, $\mathcal{B}(a) = 3$ and $\mathcal{B}(b) = \mathcal{B}(c) = 1$. In this case we write $\mathcal{A} = [a, a, b, b]$, $\mathcal{B} = [a, a, a, b, c]$ and one has $\mathcal{A} \cap \mathcal{B} = [a, a, b]$, $\mathcal{A} \cup \mathcal{B} = [a, a, a, b, b, c]$, $\mathcal{A} \oplus \mathcal{B} = [a, a, a, a, a, b, b, b, c]$, $\mathcal{A} \ominus \mathcal{B} = [b]$, and $\mathcal{B} \ominus \mathcal{A} = [a, c]$. Another notation style for writing multisets is the formal sum notation used in structures like group rings, monoid rings, group modules, and G -set modules [44, 51, 52]. Thus, one can write

$\mathcal{A} = 2a + 2b$, $\mathcal{B} = 3a + b + c$ and $\mathcal{A} \oplus \mathcal{B} = 5a + 3b + c$, and in general $C = \sum n_x x$ for a multiset C , where $n_x \in \mathbb{N}$, $x \in X$.

The **support** of a multiset $\mathcal{A}$ is defined to be the set

$$|\mathcal{A}| := \{x \in X : \mathcal{A}(x) > 0\}.$$

Even though a multiset would no longer correspond to the same object when viewed as a function if we change its domain, for the sake of practicality, for an $\mathcal{A} \in X$, we use the notation $\mathcal{A} \in A$ without hesitation, if $|\mathcal{A}| \subseteq A$ holds.

A multiset is called **finite**, if its support is a finite set.

For a multiset $\mathcal{A} : X \rightarrow \mathbb{N}$ and $k \in \mathbb{Z}^+$, we define the **k -restriction** $|\mathcal{A}|_k$ of $\mathcal{A}$ by

$$(|\mathcal{A}|_k)(x) := \min(\mathcal{A}(x), k).$$

Hence for the multiset $C = [a, b, b, d, d, d, d, d, e, e, e]$, we have the support $|C| = \{a, b, d, e\}$ and the k -restrictions $|C|_1 = [a, b, d, e]$, $|C|_2 = [a, b, b, d, d, e, e]$, $|C|_3 = [a, b, b, b, d, d, d, e, e, e]$, and so on. For practical reasons, one can disregard the formal distinction between $|C|$ and $|C|_1$, and simply write $|C|$ instead of $|C|_1$. Clearly $|C|_1 \subset |C|_2 \subset |C|_3 \cdots$, and $C = \bigcup_{k \in \mathbb{Z}^+} |C|_k$.

For multisets $\mathcal{A}$ and $\mathcal{B}$ over a universe X , we call $\mathcal{A}$ a **k -restricted submultiset** of $\mathcal{B}$, if $|\mathcal{A}|_k \subset \mathcal{B}$, and denote this by $\mathcal{A} \subset^k \mathcal{B}$. That is,

$$\mathcal{A} \subset^k \mathcal{B} \iff |\mathcal{A}|_k \subset \mathcal{B}$$

or in other words,

$$\mathcal{A} \subset^k \mathcal{B} \iff \forall i \in \{1, 2, \dots, k\} (x \in^i \mathcal{A} \implies x \in^i \mathcal{B}).$$

To note some properties that could be used later, we highlight a series of facts below that are actually quite simple and do not require proof.

Proposition 1. *The following are true for multisets $\mathcal{A}$, $\mathcal{B}$ and C over X and $k, \ell \in \mathbb{Z}^+$.*

1. $\mathcal{A} \subset \mathcal{B}$ implies $\mathcal{A} \subset^k \mathcal{B}$.
2. If $k \leq \ell$, then $\mathcal{A} \subset^\ell \mathcal{B}$ implies $\mathcal{A} \subset^k \mathcal{B}$.
3. $\mathcal{A} \subset^k \mathcal{B} \subset^k C$ implies $\mathcal{A} \subset^k C$.

A **multipoint** is a special kind of multiset with all entries are the same, like $[a]$, $[a, a]$, $[a, a, a]$, $\dots$. For $k \in \mathbb{Z}^+$, we denote the multipoint having exactly k instances of a by $[a]_k$. Hence

$$a \in^k \mathcal{A} \iff [a]_k \subset \mathcal{A}.$$

For a set A , we denote the multiset consisting of the union of all multipoints $[a]_k$ for a given $k \in \mathbb{Z}^+$ by $A^{(k)}$, that is

$$A^{(k)} = \bigcup_{a \in A} [a]_k.$$

In this case, for instance, if $A = \{a, d, e\}$, then $A^{(1)} = [a, d, e]$, $A^{(2)} = [a, a, d, d, e, e]$, and $A^{(3)} = [a, a, a, d, d, d, e, e, e]$.

A **string** $\mathcal{a}$ of length $n \in \mathbb{N}$ on a set X (called the alphabet) is an ordered array $\langle a_1, a_2, \dots, a_n \rangle$ consisting of n not necessarily distinct elements of X . If $n > 0$, then $\mathcal{a}$ is called **nonempty**. We denote the **length** of a string $\mathcal{a}$ by $n_{\mathcal{a}}$ so that $\mathcal{a} = \langle a_1, \dots, a_{n_{\mathcal{a}}} \rangle$ for a nonempty string $\mathcal{a}$.

Throughout the text, we consistently use the formal script lowercase letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$, when referring to nonempty strings, and we denote their components with the corresponding italics, like $\mathfrak{a} = \langle a_1, \dots, a_{n_{\mathfrak{a}}} \rangle$, $\mathfrak{b} = \langle b_1, \dots, b_{n_{\mathfrak{b}}} \rangle$, $\mathfrak{c} = \langle c_1, \dots, c_{n_{\mathfrak{c}}} \rangle$, and so on. On the other hand, we denote the multisets with calligraphic capitals $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$.

3 Multinumbers

We call a nonempty string $\mathfrak{a}$ of real numbers a **multinumber**, if $a_1 \leq a_2 \leq \dots \leq a_{n_{\mathfrak{a}}}$. In other words, multinumbers are nondecreasing nonempty finite sequences of real numbers.

A multinumber is called a **multiinteger**, if all of its terms are integers, and a multinumber $\mathfrak{a}$ is called **nonnegative**, if $\{a_1, \dots, a_{n_{\mathfrak{a}}}\} \subseteq \mathbb{R}_0^+$, or equivalently $a_1 \geq 0$, and it is called **positive** if $\{a_1, \dots, a_{n_{\mathfrak{a}}}\} \subseteq \mathbb{R}^+$, or equivalently $a_1 > 0$.

One can characterize the real number 1 over the entire set of real numbers as follows: it is the unique number for which the proposition "1 is less than x " is false for all numbers $x < 1$ and true for all $x > 1$. In this context, we can interpret a multinumber $\mathfrak{a} = \langle 3, 5, 7, 7, 7, 11, 11 \rangle$ as follows: the statement " $\mathfrak{a}$ is less than x " is false for all real numbers $x < 3$, it is true with a truth degree of no more than 1 for $x < 5$, it is 2-fold true for $x < 7$ with a truth degree not exceeding 2, it is 5-fold true for $x < 11$ with a truth degree not exceeding 5, and it is 7-fold true for $x > 11$ but with a truth degree not exceeding 7. This multinumber can thus be understood as a generalized quantity representing multiple layers of truth.

We denote the set of all multinumbers by $\mathbb{M}$, the set of all positive multinumbers by $\mathbb{M}^+$, and the set of all nonnegative multinumbers by $\mathbb{M}_0^+$. $\mathbb{M}$ is partially ordered by the relation

$$\mathfrak{a} \preccurlyeq \mathfrak{b} : \Longleftrightarrow n_{\mathfrak{a}} \geq n_{\mathfrak{b}} \text{ and } a_i \leq b_i \text{ for all } 1 \leq i \leq n_{\mathfrak{b}}. \quad (1)$$

As expected, we write $\mathfrak{a} \prec \mathfrak{b}$ in the case that $\mathfrak{a} \preccurlyeq \mathfrak{b}$, $\mathfrak{a} \neq \mathfrak{b}$. Moreover, $(\mathbb{M}, \preccurlyeq)$ has a lattice structure. The suprema for two-element multisets are given by

$$\mathfrak{a} \vee \mathfrak{b} := \langle a_1 \vee b_1, \dots, a_{n_{\mathfrak{a}} \wedge n_{\mathfrak{b}}} \vee b_{n_{\mathfrak{a}} \wedge n_{\mathfrak{b}}} \rangle,$$

where $\vee$ and $\wedge$ respectively stand for the supremum and infimum on real numbers. Without loss of generality, if we suppose that $n_{\mathfrak{a}} \leq n_{\mathfrak{b}}$, then the infimum of the multiset $[\mathfrak{a}, \mathfrak{b}]$ of strings is given by

$$\mathfrak{a} \wedge \mathfrak{b} := \langle a_1 \wedge b_1, \dots, a_{n_{\mathfrak{a}}} \wedge b_{n_{\mathfrak{a}}}, b_{n_{\mathfrak{a}}+1}, \dots, b_{n_{\mathfrak{b}}} \rangle.$$

We define the sum of two multinumbers as

$$\mathfrak{a} \boxplus \mathfrak{b} := \langle a_1 + b_1, \dots, a_{n_{\mathfrak{a}} \wedge n_{\mathfrak{b}}} + b_{n_{\mathfrak{a}} \wedge n_{\mathfrak{b}}} \rangle.$$

Hence $n_{\mathfrak{a} \boxplus \mathfrak{b}} = \min(n_{\mathfrak{a}}, n_{\mathfrak{b}})$ and one can write $(a + b)_i := a_i + b_i$ for all $1 \leq i \leq n_{\mathfrak{a} \boxplus \mathfrak{b}}$. This multinumber addition is clearly associative and commutative, while it has no unit. This makes $(\mathbb{M}, \boxplus)$ a semigroup, and $\mathbb{M}_0^+$ is a subsemigroup of $\mathbb{M}$. In fact, $(\mathbb{M}, \boxplus)$ has a generalized group structure in the sense of [40], where the partial identity for each $\mathfrak{a} \in \mathbb{M}$ is the string of length $n_{\mathfrak{a}}$ with all entries equal to 0.

We provide some simple examples to illustrate our basic notations about multinumbers: $\langle 0, \sqrt{2}, 5 \rangle \prec \langle 1, 3, 7 \rangle \prec \langle 1, \pi \rangle \prec \langle 1, 4 \rangle \prec \langle 3 \rangle$, but $\langle 2, 3, 5 \rangle \not\prec \langle 1, 4 \rangle$, $\langle 0, 1 \rangle \not\prec \langle 1, 3, 5 \rangle$ and $\langle 3, 11 \rangle \not\prec \langle 4, 4 \rangle$. $\langle 3, 6, 7, 7, 9 \rangle \vee \langle 4, 5, 5 \rangle = \langle 4, 6, 7 \rangle$, $\langle 3, 6, 7, 7, 9 \rangle \wedge \langle 4, 5, 5 \rangle = \langle 3, 5, 5, 7, 9 \rangle$, and $\langle 3, 6, 7, 7, 9 \rangle \boxplus \langle 4, 5, 5 \rangle = \langle 7, 11, 12 \rangle$.

Note that each multinumber is uniquely associated to a finite multiset over $\mathbb{R}$. Collecting the terms of a

multinumber together gives a finite multiset, and each finite multiset corresponds to a unique multinumber, because once a finite number of instances of real numbers are given, there is essentially a unique way to order them nondecreasingly.

Another way to think about multinumbers is to view them as nondecreasing extended real number sequences where all terms, except for the first few, are ∞ . This perspective is particularly useful when it comes to understanding ordering, lattice structure, and addition. For instance, to think $\langle 0, 1 \rangle$ as $\langle 0, 1, \infty, \infty, \infty, \dots \rangle$, and $\langle 1, 3, 5 \rangle$ as $\langle 1, 3, 5, \infty, \infty, \dots \rangle$ makes easier to observe the facts $\langle 0, 1 \rangle \not\preceq \langle 1, 3, 5 \rangle$ and $\langle 0, 1 \rangle \boxplus \langle 1, 3, 5 \rangle = \langle 0, 2 \rangle$.

The lattice $(\mathbb{M}, \preceq)$ is not complete. To observe this, for instance consider the subset $\mathcal{A} = \{\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \dots\}$, which has no supremum, and also the subset $\mathcal{B} = \{\langle 0 \rangle, \langle 0, 0 \rangle, \langle 0, 0, 0 \rangle, \langle 0, 0, 0, 0 \rangle, \dots\}$ with no infimum.

Proposition 2. $(\mathbb{M}, \boxplus, \preceq)$ is an ordered semigroup.

Proof. Since the semigroup $(\mathbb{M}, \boxplus)$ is commutative, it is enough to show that $a \preceq b$ implies $a \boxplus c \preceq b \boxplus c$ for all $a, b, c \in \mathbb{M}$.

Suppose that $a \preceq b$. Then $n_a \geq n_b$ and $a_i \leq b_i$ for all $1 \leq i \leq n_b$. In this case $n_{a \boxplus c} = n_a \wedge n_c \geq n_b \wedge n_c = n_{b \boxplus c}$ and by $n_{b \boxplus c} \leq n_b$, we have $a_i + c_i \leq b_i + c_i$ for all $1 \leq i \leq n_{b \boxplus c}$, which requires $a \boxplus c \preceq b \boxplus c$. $\square$

Remark 1. While the implication $a \preceq b \implies a \boxplus c \preceq b \boxplus c$ always holds by Proposition 2, it is not true in general that $a \prec b \implies a \boxplus c \prec b \boxplus c$. As a simple counterexample, consider the multinumbers $a = \langle 1, 4, 7 \rangle$, $b = \langle 1, 4, 9 \rangle$ and $c = \langle 2, 3 \rangle$. We have $a \prec b$, but $a \boxplus c \not\prec b \boxplus c$ by the equality $a \boxplus c = \langle 3, 7 \rangle = b \boxplus c$.

Proposition 3. If $a \in \mathbb{M}$ and $b \in \mathbb{M}_0^+$, then $a \preceq a \boxplus b$.

Proof. Follows from $n_a \geq n_a \wedge n_b = n_{a \boxplus b}$ and $a_i \leq a_i + b_i$ by $b_i \geq 0$. $\square$

We denote the set of multinumbers of length n by $\mathbb{M}(n)$ for each $n \in \mathbb{Z}^+$, and if we restrict ourselves to nonnegative multisets of length n , we use the similar notation $\mathbb{M}_0^+(n)$. In particular $\mathbb{M}(1) = \{\langle x \rangle : x \in \mathbb{R}\}$. The addition and order on $\mathbb{M}(1)$ satisfies $\langle x \rangle \boxplus \langle y \rangle = \langle x + y \rangle$, and $\langle x \rangle \preceq \langle y \rangle \iff x \leq y$. This makes possible to think $\mathbb{R}$ as embedded into $\mathbb{M}$ as $\mathbb{M}(1)$.

Multinumbers can be multiplied by nonnegative real numbers in an obvious way. If a is a multinumber and $\lambda \geq 0$ is a real number, then the multinumber λa is defined as $\langle \lambda a_1, \dots, \lambda a_{n_a} \rangle$.

4 Multimetric and Multinormed Spaces

Thanks to the multinumbers, we introduce the notion of a multimetric below as a reinforced and graded distance measurement tool.

Definition 1. A **multimetric** $\mathcal{d}$ on a set X is a function $\mathcal{d} : X \times X \longrightarrow \mathbb{M}_0^+$ satisfying the following properties for all $x, y, z \in X$.

(MM0) $\mathcal{d}(x, y) \preceq \langle 0 \rangle$ implies $x = y$.

(MM1) $\mathcal{d}(x, x) \preceq \langle 0 \rangle$.

(MM2) $\mathcal{d}(x, y) = \mathcal{d}(y, x)$.

(MM3) $\mathcal{d}(x, z) \preceq \mathcal{d}(x, y) \boxplus \mathcal{d}(y, z)$.

In this case $(X, \mathcal{d})$ is called a **multimetric space**.

In general, we denote the multinumber $\mathcal{d}(x, y)$ as $\langle d_1(x, y), \dots, d_{n_{\mathcal{d}(x, y)}}(x, y) \rangle$, when we want to refer to the components of $\mathcal{d}(x, y)$ explicitly. We adopt the convention that when we include a term $d_i(x, y)$ in a discussion, we also assume the existence of it. For instance, if we require $d_3(x, y) < \varepsilon$ in an expression, this means that $d_3(x, y)$ actually exists, that is the string $\mathcal{d}(x, y)$ is of length at least 3, and its third term is less than ε .

In the context of (MM3), if the multinumbers $\mathcal{d}(x, y)$ and $\mathcal{d}(y, z)$ are both of length at least k , that is if both $d_k(x, y)$ and $d_k(y, z)$ exist, then also $d_k(x, z)$ exists, and the inequality $d_k(x, z) \leq d_k(x, y) + d_k(y, z)$ hold. Hence, if $(X, \mathcal{d})$ is a multimetric space, then (X, d_1) is a metric space. We call this the **metric support** of $(X, \mathcal{d})$, and write $|(X, \mathcal{d})| := (X, d_1)$.

Example 1. Every metric space (X, d) can be thought as a multimetric space $(X, \mathcal{d})$, where $\mathcal{d}(x, y) = \langle d(x, y) \rangle$ for all $x, y \in X$.

Example 2. Let $X = \{a, b, c\}$ and the values of $\mathcal{d}$ be given as the table below:

$\mathcal{d}$	a	b	c
a	$\langle 0 \rangle$	$\langle 1, 3, 7 \rangle$	$\langle 3, 7, 11, 12, 19 \rangle$
b	$\langle 1, 3, 7 \rangle$	$\langle 0, 6, 7, 9, 10, 11 \rangle$	$\langle 3, 4, 5, 5 \rangle$
c	$\langle 3, 7, 11, 12, 19 \rangle$	$\langle 3, 4, 5, 5 \rangle$	$\langle 0, 0, 5, 8 \rangle$

From this table, for instance, we can read that $\mathcal{d}(a, b) = \langle 1, 3, 7 \rangle$, $\mathcal{d}(b, c) = \langle 3, 4, 5, 5 \rangle$, and $\mathcal{d}(b, b) = \langle 0, 6, 7, 9, 10, 11 \rangle$. With these values, it is easy to observe that $(X, \mathcal{d})$ forms a multimetric space.

These kind of values can be interpreted logically in various ways. Since the length of the string $\langle 3, 7, 11, 12, 19 \rangle$ is 5, we have a five-level reinforced truth notion here. According to this, the statement that the distance between a and c is less than 2 is false. The statement that this distance is less than 4 is true, with a truth multiplicity of 1. The statement that the distance between a and c is less than 15 is also true, and since there are four values in the string less than 15, the multiplicity of this truth is 4. The interpretation of this n -fold truth can vary depending on the purpose of the application. For example, one possible interpretation could be that even if it is no longer guaranteed that the distance between the initial position of point a and the final position of point c (and vice versa) remains less than 5 after a degenerative external effect is applied to the system, it is still guaranteed to remain less than 10. However, under a slightly more destructive effect, it is no longer guaranteed to be less than 10, but it can be ensured that it remains less than or equal to 11. The proposition “distance between a and c is less than 20”, or that it has a finite value, is 5-fold true in both cases.

Example 3. Multimetric spaces can, depending on the situation, combine values from various types of generalized metric spaces. For example, let a metric d , a pseudometric ρ , a dislocated metric δ [1, 17], a partial metric p [20, 45] on X , and a bipolar metric b [13, 26, 47] on (A, B) , where $A, B \subseteq X$ are nonempty, be given. For brevity, adopt the shorthand functional notation $(f + g)(x, y) := f(x, y) + g(x, y)$. Say $d_1(x, y) := d(x, y)$, $d_2(x, y) := (d + \rho)(x, y)$, $d_3(x, y) := (d + \rho + \delta)(x, y)$ and $d_4(x, y) := (d + \rho + \delta + p)(x, y)$ for all $x, y \in X$, and

$$d_5(x, y) := \begin{cases} (d + \rho + \delta + p + b)(x, y), & \text{if } (x, y) \in A \times B \\ (d + \rho + \delta + p)(x, y) + b(y, x), & \text{if } (x, y) \in B \times A \end{cases}$$

for all $(x, y) \in (A \times B) \cup (B \times A)$. In particular, d_5 is well-defined as $b(x, y) = b(y, x)$ whenever $x, y \in A \cap B$.

Then, the function $\mathcal{d} : X \times X \longrightarrow \mathbb{M}_0^+$,

$$\mathcal{d}(x, y) = \begin{cases} \langle d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y), d_5(x, y) \rangle, & (x, y) \in (A \times B) \cup (B \times A) \\ \langle d_1(x, y), d_2(x, y), d_3(x, y), d_4(x, y) \rangle, & \text{otherwise} \end{cases}$$

defines a multimetric on X .

Example 4. Let (X, Ω) be an octonion-valued metric space (see [14]), and define

$$\mathcal{d}(x, y) = \langle o_0, o_0 + o_1, o_0 + o_1 + o_2, \dots, o_0 + o_1 + o_2 + o_3 + o_4 + o_5 + o_6 + o_7 \rangle,$$

where $\Omega(x, y) = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7$. Then $(X, \mathcal{d})$ becomes a multimetric space.

Example 5. Below are the approximate total round-trip transportation times between eight cities, using various modes of transport. The first values in the strings represent the estimated round-trip flight time between two cities. If present, the second value in the string represents the approximate round-trip time by road. Finally, when there is a third value, it indicates that rail travel is available, and the value represents the estimated round-trip travel time by train. These values can be modelled as a multimetric space structure, whenever they satisfy the triangle inequality for multinumbers.

$\mathcal{d}$	Detroit	Doha	Houston	Istanbul	Lagos	Madrid	Sydney	Toronto
Detroit	$\langle 0, 0, 0 \rangle$	$\langle 26 \rangle$	$\langle 7, 40 \rangle$	$\langle 20 \rangle$	$\langle 20 \rangle$	$\langle 16 \rangle$	$\langle 40 \rangle$	$\langle 2, 8, 9 \rangle$
Doha	$\langle 26 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 28 \rangle$	$\langle 9, 80 \rangle$	$\langle 18, 300 \rangle$	$\langle 14, 120 \rangle$	$\langle 28 \rangle$	$\langle 26 \rangle$
Houston	$\langle 7, 40 \rangle$	$\langle 28 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 23 \rangle$	$\langle 24 \rangle$	$\langle 20 \rangle$	$\langle 34 \rangle$	$\langle 8, 44 \rangle$
Istanbul	$\langle 20 \rangle$	$\langle 9, 80 \rangle$	$\langle 23 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 13, 180 \rangle$	$\langle 9, 60, 90 \rangle$	$\langle 32 \rangle$	$\langle 21 \rangle$
Lagos	$\langle 20 \rangle$	$\langle 18, 300 \rangle$	$\langle 24 \rangle$	$\langle 13, 180 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 11, 200 \rangle$	$\langle 36 \rangle$	$\langle 22 \rangle$
Madrid	$\langle 16 \rangle$	$\langle 14, 120 \rangle$	$\langle 20 \rangle$	$\langle 9, 60, 90 \rangle$	$\langle 11, 200 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 34 \rangle$	$\langle 16 \rangle$
Sydney	$\langle 40 \rangle$	$\langle 28 \rangle$	$\langle 34 \rangle$	$\langle 32 \rangle$	$\langle 36 \rangle$	$\langle 34 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 38 \rangle$
Toronto	$\langle 2, 8, 9 \rangle$	$\langle 26 \rangle$	$\langle 8, 44 \rangle$	$\langle 21 \rangle$	$\langle 22 \rangle$	$\langle 16 \rangle$	$\langle 38 \rangle$	$\langle 0, 0, 0 \rangle$

Example 6. Every positive natural number can be uniquely represented by a sequence based on prime numbers, where the first term gives the multiplicity of 2, the second term gives the multiplicity of 3, the third term gives that of 5, and in general, the n th term represents the multiplicity of the n th prime number. For example, 12, 27, 40, and 96 are represented by the sequences $(2, 1, 0, 0, \dots)$, $(0, 3, 0, 0, \dots)$, $(3, 0, 1, 0, 0, \dots)$, and $(5, 1, 0, 0, \dots)$, respectively, with trailing zeros. Suppose that such representations of two given numbers $x, y \in \mathbb{Z}^+$ are the sequences (x_n) and (y_n) . If $x = y$, define $\mathcal{d}(x, y) = \langle 0 \rangle$. If $x \neq y$, define $\mathcal{d}(x, y)$ as the multinumber corresponding to the multiset of non-zero absolute differences $|x_n - y_n|$. For example, $\mathcal{d}(12, 27) = \langle 2, 2 \rangle$, $\mathcal{d}(12, 40) = \langle 1, 1, 1 \rangle$, $\mathcal{d}(12, 96) = \langle 3 \rangle$, $\mathcal{d}(27, 40) = \langle 1, 3, 3 \rangle$, $\mathcal{d}(27, 96) = \langle 2, 5 \rangle$, and $\mathcal{d}(40, 96) = \langle 1, 1, 2 \rangle$. This gives a multimetric space $(\mathbb{Z}^+, \mathcal{d})$.

Example 7. The approach from the previous example can be easily adapted for the set $\mathbb{R}^\infty$ consisting of real sequences that eventually terminate in zeros. Accordingly, for the elements $x = (3, 1, \frac{1}{2}, 0, 0, \dots)$ and $y = (\frac{9}{2}, 4, -1, 0, \frac{1}{3}, 0, 0, \dots)$ in $\mathbb{R}^\infty$, we have $\mathcal{d}(x, y) = \langle \frac{1}{3}, \frac{3}{2}, \frac{3}{2}, 3 \rangle$. This defines a multimetric space $(\mathbb{R}^\infty, \mathcal{d})$.

Example 8. $\mathcal{d}(x, y) = \langle |x - y|, |x| + |y| \rangle$ gives a multimetric on $\mathbb{R}$.

Now, we explain the concept of a multinorm that can be defined on a vector space.

Definition 2. Let V be a vector space on $\mathbb{R}$ or $\mathbb{C}$, with the zero vector $\vec{0}$. Then a function $\|\cdot\| : V \longrightarrow \mathbb{M}_0^+$ is called a **multinorm** on V , if the following are satisfied for all $\lambda \in \mathbb{R}$ and $\vec{u}, \vec{v} \in V$.

$$(MN0) \|\vec{v}\| \preccurlyeq \langle 0 \rangle \text{ implies } x = \vec{0}.$$

$$(MN1) \|\vec{0}\| \preccurlyeq \langle 0 \rangle.$$

$$(MN2) \|\lambda \vec{v}\| = |\lambda| \|\vec{v}\|.$$

$$(MN3) \|\vec{u} + \vec{v}\| \preccurlyeq \|\vec{u}\| \boxplus \|\vec{v}\|.$$

In this case we call the pair $(V, \|\cdot\|)$ a **multinormed space**.

As in the classical case, there is a clear and self-evident relationship between multinormed spaces and multimetric spaces, which is stated in the following without proof.

Theorem 1. If $(V, \|\cdot\|)$ is a multinormed space, then the function $\mathcal{d}_{\|\cdot\|} : V \times V \longrightarrow \mathbb{M}_0^+$ defined by $\mathcal{d}_{\|\cdot\|}(x, y) := \|x - y\|$ for all $x, y \in V$, is a multimetric on V .

5 Topological Structures on Multimetric Spaces

To illustrate how topological concepts fit within the framework of multimetric spaces, we will explore some basic topological concepts. There are undoubtedly multiple ways to develop topological notions in multimetric spaces. For example, one could define open balls by selecting radii as nonnegative multinumbers while taking points as centers. This approach would yield balls as sets, however, in some cases this can cause a dislocated ball, that is a ball not including its center. Another approach, instead, takes radii as nonnegative real numbers, as is customary, but since the distances are expressed in terms of multinumbers, this will give us multisets rather than ordinary sets. As we will later discuss, even this particular approach allows for two major distinct interpretations. However, before delving into these discussions, let us first define some specific subsets and submultisets in multimetric spaces.

Definition 3. Given a multimetric space $(X, \mathcal{d})$, and let $x \in X$, $r \in \mathbb{M}^+$. The **ball**, the **disk**, and the **sphere** with center x and radius r are defined to be the subsets of X respectively given by

$$\mathcal{B}(x, r) := \{a \in X : \mathcal{d}(x, a) \prec r\},$$

$$\mathcal{D}(x, r) := \{a \in X : \mathcal{d}(x, a) \preccurlyeq r\},$$

$$\mathcal{S}(x, r) := \mathcal{D}(x, r) \setminus \mathcal{B}(x, r).$$

Through these sets, X acquires a topological structure in the classical sense, which will be explained in Subsection 5.1.

Example 9. Consider the multimetric space $(X, \mathcal{d})$ in Example 2 given with a table.

Since $\mathcal{d}(c, a) = \langle 3, 7, 11, 12, 19 \rangle \not\preccurlyeq \langle \pi, \sqrt{19}, 2\pi, 7 \rangle$, $\mathcal{d}(c, b) = \langle 3, 4, 5, 5 \rangle \preccurlyeq \langle \pi, \sqrt{19}, 2\pi, 7 \rangle$, and $\mathcal{d}(c, c) = \langle 0, 0, 5, 8 \rangle \not\preccurlyeq \langle \pi, \sqrt{19}, 2\pi, 7 \rangle$, we have $\mathcal{B}(c, \langle \pi, \sqrt{19}, 2\pi, 7 \rangle) = \{b\}$. Note especially that this ball is dislocated, in the sense that $c \notin \mathcal{B}(c, \langle \pi, \sqrt{19}, 2\pi, 7 \rangle)$. Similarly we have $\mathcal{D}(c, \langle \pi, \sqrt{19}, 2\pi, 7 \rangle) = \{b\}$ and $\mathcal{S}(c, \langle \pi, \sqrt{19}, 2\pi, 7 \rangle) = \emptyset$.

Now, we move to the approach where the radius is a positive real number rather than a positive multinumber. In this case, radii lose their “multi”ness, while the sets gain it; meaning that the radii are no longer multinumbers, but the resulting collections are now multisets.

Definition 4. Given a multimetric space $(X, \mathcal{d})$, and let $x \in X$, $r > 0$. The **multiball** $\mathbb{B}(x, r)$, the **multidisk** $\mathbb{D}(x, r)$, and the **multisphere** $\mathbb{S}(x, r)$ with center x and radius r are defined to be the multisets over X given by

$$\begin{aligned}\mathbb{B}(x, r)(y) &:= \sup\{i \in \mathbb{N} : 0 \leq i \leq n_{\mathcal{d}(x,y)} \text{ and } d_i(x, y) < r\}, \\ \mathbb{D}(x, r)(y) &:= \sup\{i \in \mathbb{N} : 0 \leq i \leq n_{\mathcal{d}(x,y)} \text{ and } d_i(x, y) \leq r\}, \\ \mathbb{S}(x, r) &:= \mathbb{D}(x, r) \ominus \mathbb{B}(x, r).\end{aligned}$$

Note especially that, if $\{i \in \mathbb{N} : 0 \leq i \leq n_{\mathcal{d}(x,y)} \text{ and } d_i(x, y) < r\} = \emptyset$ then $\mathbb{B}(x, r)(y) = 0$, and if $\{i : 0 \leq i \leq n_{\mathcal{d}(x,y)} \text{ and } d_i(x, y) \leq r\} = \emptyset$ then $\mathbb{D}(x, r)(y) = 0$, since the suprema are taken over the lattice $\mathbb{N}$ having 0 as the smallest element.

Clearly $\mathbb{B}(x, r) \subset \mathbb{D}(x, r)$ and $\mathbb{S}(x, r) \subset \mathbb{D}(x, r)$ for all $r > 0$, and $0 < s < r$ implies $\mathbb{B}(x, s) \subset \mathbb{B}(x, r)$ and $\mathbb{D}(x, s) \subset \mathbb{D}(x, r)$.

Example 10. Consider the multimetric space $(X, \mathcal{d})$ in Example 2.

We describe the multidisk $\mathbb{D}(c, 5)$. Since $\mathcal{d}(c, a) = \langle 3, 7, 11, 12, 19 \rangle$ and $d_1(c, a) = 3 \leq 5$, but other terms of this string are greater than 5, thus the multiset $\mathbb{D}(c, 5)$ contains exactly one instance of a . On the other hand $\mathcal{d}(c, b) = \langle 3, 4, 5, 5 \rangle$ and all four terms are less than or equal to 5, so that $\mathbb{D}(c, 5)$ has four instances of b . Lastly since $\mathcal{d}(c, c) = \langle 0, 0, 5, 8 \rangle$ has only three terms not exceeding 5, $\mathbb{D}(c, 5)$ has three instances of c . Hence we have found $\mathbb{D}(c, 5) = [a, b, b, b, b, c, c, c]$.

With a similar approach, one has the multiball $\mathbb{B}(c, 5) = [a, b, b, c, c]$, and the multisphere $\mathbb{S}(c, 5) = \mathbb{D}(c, 5) \ominus \mathbb{B}(c, 5) = [b, b, c]$.

We now present some basic properties of multiballs, limited to those that will be used in the text.

Proposition 4. Let $(X, \mathcal{d})$ be a multimetric space and let $x \in X$, $\varepsilon > 0$.

1. $x \in \mathbb{B}(x, \varepsilon)$.
2. If $\varepsilon < \delta$, then $\mathbb{B}(x, \varepsilon) \subset \mathbb{B}(x, \delta)$.
3. $\mathbb{B}(x, \delta) \cap \mathbb{B}(x, \varepsilon) = \mathbb{B}(x, \min(\delta, \varepsilon))$.

Proof. a) By $\mathcal{d}(x, x) \preceq 0$, necessarily $d_1(x, x) = 0 < \varepsilon$, hence $\mathbb{B}(x, \varepsilon)$ has at least one instance of x in it.

b) If $y \in^k \mathbb{B}(x, \varepsilon)$, then $d_k(x, y) < \varepsilon < \delta$, which means that $y \in^k \mathbb{B}(x, \delta)$.

c) $\mathbb{B}(x, \min(\delta, \varepsilon)) \subset \mathbb{B}(x, \delta) \cap \mathbb{B}(x, \varepsilon)$ follows from 2. Conversely, if $y \in^k \mathbb{B}(x, \delta) \cap \mathbb{B}(x, \varepsilon)$, then $d_k(x, y) < \delta$ and $d_k(x, y) < \varepsilon$, so that $d_k(x, y) < \min(\delta, \varepsilon)$ and $y \in^k \mathbb{B}(x, \min(\delta, \varepsilon))$. $\square$

Multiballs can be applied to topological concepts in at least two distinct ways. In Subsection 5.2, the first approach to be adopted is to treat multiballs as the building blocks of openness. Since multiballs are multisets, this leads to concepts like openness and closedness are defined for submultisets. On the other hand, in Subsection 5.3, our alternative approach is to assign multiplicities to concepts like openness and closedness, to work with n -fold versions of these. In this case, we are not directly dealing with multisets, but instead, the concepts have been “multiplied” rather than the elements in a set, to reflect the nature of multimetric spaces.

5.1 Topology

We demonstrate how a topological structure can be given to a multimetric space through the use of balls. Since other topological concepts related to this approach will directly follow from this structure, we refrain from much detail.

Definition 5. Given a multimetric space $(X, \mathcal{d})$. We call a subset A of X **open**, if for all $x \in A$, there exists some $r \in \mathbb{M}^+$ such that $x \in \mathcal{B}(x, r) \subseteq A$.

The condition that $\mathcal{B}(x, r)$ must contain x , ensures that dislocated neighborhoods are avoided. Now, without delay, we prove that this definition gives rise to a topology on X .

Theorem 2. Let $(X, \mathcal{d})$ be a multimetric space, and τ be the family of all open subsets of X . Then,

- a) $X, \emptyset \in \tau$.
- b) $\{A_i\}_{i \in I} \subseteq \tau \implies \bigcup_{i \in I} A_i \in \tau$.
- c) $A, B \in \tau \implies A \cap B \in \tau$.

Proof. a) Follows from the definition of an open subset.

b) If $x \in \bigcup_{i \in I} A_i \in \tau$, then $x \in A_{i_0}$ for some $i_0 \in I$, and by openness of A_{i_0} , there exists an $r_{i_0} \in \mathbb{M}^+$ such that $x \in \mathcal{B}(x, r_{i_0}) \subseteq A_{i_0} \subseteq \bigcup_{i \in I} A_i$.

c) If $A, B \in \tau$ and $x \in A \cap B$, then $x \in \mathcal{B}(x, r) \subseteq A$ and $x \in \mathcal{B}(x, s) \subseteq B$ for some $r, s \in \mathbb{M}^+$. In this case, $r_1, s_1 > 0$, and so $r_1 \wedge s_1 > 0$, that is $r \wedge s \in \mathbb{M}^+$. Then, $\mathcal{d}(x, x) \prec r$ and $\mathcal{d}(x, x) \prec s$ together give $\mathcal{d}(x, x) \prec r \wedge s$, and since $r \wedge s \preceq r$ and $r \wedge s \preceq s$, we have $x \in \mathcal{B}(x, r \wedge s) \subseteq \mathcal{B}(x, r) \cap \mathcal{B}(x, s) \subseteq A \cap B$. $\square$

According to this result, multimetric spaces have the ability to generate a nontrivial topology in the classical sense. On the other hand, we know that every multimetric space, generates a metric space called the metric support, with the first components of multinumber values of the distance function, and of course, this metric induces a topology as well. Now, we present a result that explains the fine-coarse relationship between these two topologies.

Theorem 3. Let $\tau_{\mathcal{d}}$ be the topology on a set X , induced by a multimetric space $(X, \mathcal{d})$ as described in Theorem 2, and τ_{d_1} be the topology induced by the metric support (X, d_1) . Then $\tau_{\mathcal{d}}$ is finer than τ_{d_1} .

Proof. Let A be an open set in the topological space (X, τ_{d_1}) and $x \in A$. Then there exists an open ball $B(x, r)$ in the classical sense with respect to the metric space (X, d_1) , such that $B(x, r) \subseteq A$. Consider the multinumber $r = \langle r \rangle$ of length 1, and the ball $\mathcal{B}(x, r)$. $\langle 0 \rangle \prec \langle r \rangle$ since $r > 0$, and $\mathcal{d}(x, x) \preceq \langle 0 \rangle$ by (MM1). Hence $\mathcal{d}(x, x) \prec \langle r \rangle$ so that $x \in \mathcal{B}(x, r)$. Now, if $y \in \mathcal{B}(x, r)$, then $\mathcal{d}(x, y) \prec r$, which in particular implies $d_1(x, y) \leq r$ and $y \in B(x, r)$. Therefore $x \in \mathcal{B}(x, r) \subseteq B(x, r) \subseteq A$, which means that A is also open in the topological space $(X, \tau_{\mathcal{d}})$. $\square$

Although we have shown that the topology corresponding to a multimetric space is finer than the topology generated by its corresponding metric support, the question of whether these two topologies are equal or not, remains open. The following example demonstrates that such equality is not necessarily required.

Example 11. Consider the multimetric $\mathcal{d}$ given by

$$\mathcal{d}(x, y) = \begin{cases} \langle |x - y| \rangle, & \text{if } x \neq y. \\ \langle 0, 0 \rangle, & \text{if } x = y. \end{cases}$$

for all $x, y \in \mathbb{R}$. Then d_1 is the usual metric on $\mathbb{R}$, and it induces the usual topology. On the other hand, for any $x \in \mathbb{R}$ and $r = \langle 1, 1 \rangle$, $\mathcal{d}(x, x) = \langle 0, 0 \rangle \prec \langle 1, 1 \rangle$, but if $x \neq y$, $\langle |x - y| \rangle \not\prec \langle 1, 1 \rangle$ by Equation 1, since $n_{\langle d(x, y) \rangle} = 1 \not\geq 2 = n_{\langle 1, 1 \rangle}$. Hence $\mathcal{B}(x, \langle 1, 1 \rangle) = \{x\}$, and $\mathcal{d}$ induces the discrete topology on $\mathbb{R}$.

5.2 Multiset Topology

Throughout this subsection, we suppose that I is an index set, and $\mathcal{A}, \mathcal{A}_i, \mathcal{B} \in X$ for all $i \in I$. Before addressing its relevance to multimetric spaces, we first highlight a topology-like structure defined on the class of all multisets over a given set. While the concept of multiset topology has been studied in [21], here we provide a slightly revised definition to suit our purposes.

Definition 6. A **multiset topology** on a set X is a family $\mathcal{T} \subseteq \mathcal{M}(X)$ such that

- a) $\emptyset \in \mathcal{T}$ and $X^{(k)} \in \mathcal{T}$ for all $k \in \mathbb{Z}^+$.
- b) If $\mathcal{A} \in X$, $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ and $\{\mathcal{A}_i\}_{i \in I} \subseteq \mathcal{T}$, then $\mathcal{A} \in \mathcal{T}$.
- c) If $\mathcal{A}, \mathcal{B} \in \mathcal{T}$, then $\mathcal{A} \cap \mathcal{B} \in \mathcal{T}$.

In this case, the pair $(X, \mathcal{T})$ is called a **multiset topological space**.

Specifically, condition (b) is written in an indirect manner due to the fact that an arbitrary union of multisets does not necessarily result in a multiset, as explained in Section ?? regarding the arbitrary union notation.

Definition 7. In a multimetric space $(X, \mathcal{d})$, a multiset $\mathcal{A} \in X$ is called an **open multiset**, if for all $x \in^k \mathcal{A}$, there exists an $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subset^k \mathcal{A}$.

With this definition, clearly each multiball $\mathbb{B}(x, \varepsilon)$ is automatically an open multiset.

Theorem 4. Let $(X, \mathcal{d})$ be a multimetric space. Then the family $\mathcal{T}$ of all open multisets over X , is a multiset topology over X .

Proof. By the definition of an open multiset, the multiset $\emptyset$ is necessarily open, as it does not have any element.

Consider the set $X^{(k)}$, $k \in \mathbb{Z}^+$. $X^{(k)}$ has exactly k instances of each element in X . On the other hand, by definition, the k -restrictions $|\mathbb{B}(x, \varepsilon)|_k$ have at most k instances of elements. Thus $|\mathbb{B}(x, \varepsilon)|_k \subset X^{(k)}$, which means that $\mathbb{B}(x, \varepsilon) \subset^k X^{(k)}$, and the multiset $X^{(k)}$ is open.

Now suppose that $\mathcal{A} \in X$, $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$ and $\{\mathcal{A}_i\}_{i \in I} \subseteq \mathcal{T}$. If $x \in^k \bigcup_{i \in I} \mathcal{A}_i$, then $x \in^k \mathcal{A}_{i_0}$ for some $i_0 \in I$, and since $\mathcal{A}_{i_0} \in \mathcal{T}$, there is an $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subset^k \mathcal{A}_{i_0} \subset \bigcup_{i \in I} \mathcal{A}_i$, which implies $\mathbb{B}(x, \varepsilon) \subset^k \bigcup_{i \in I} \mathcal{A}_i$.

Lastly, if $\mathcal{A}, \mathcal{B} \in \mathcal{T}$ and $x \in^k \mathcal{A} \cap \mathcal{B}$, then there are $\varepsilon, \delta > 0$ such that $\mathbb{B}(x, \varepsilon) \subset^k \mathcal{A}$ and $\mathbb{B}(x, \delta) \subset^k \mathcal{B}$. Then $\mathbb{B}(x, \min(\varepsilon, \delta)) \subset \mathbb{B}(x, \varepsilon) \cap \mathbb{B}(x, \delta) \subset^k \mathcal{A} \cap \mathcal{B}$ by Proposition 1 1 and 3, and Proposition 4 2. $\square$

Theorem 5. Multiballs are open multisets with respect to the multiset topology.

Proof. Consider a multiball $\mathbb{B}(x, \varepsilon)$ in a multimetric space $(X, \mathcal{d})$, and suppose $y \in^k \mathbb{B}(x, \varepsilon)$. Then $d_k(x, y) < \varepsilon$. Say $\delta := \varepsilon - d_k(x, y)$.

We prove that $\mathbb{B}(y, \delta) \subset^k \mathbb{B}(x, \varepsilon)$, or equivalently $|\mathbb{B}(y, \delta)|_k \subset \mathbb{B}(x, \varepsilon)$. Let $z \in^\ell |\mathbb{B}(y, \delta)|_k$. By the definition of a k -restriction, $|\mathbb{B}(y, \delta)|_k$ can include at most k instances of an element, so that $\ell \leq k$, and consequently $d_\ell(x, y) \leq d_k(x, y)$. Also $z \in^\ell |\mathbb{B}(y, \delta)|_k \subset \mathbb{B}(y, \delta)$ gives $d_\ell(y, z) < \delta$. Then $d_\ell(x, z) \leq d_\ell(x, y) + d_\ell(y, z) < d_k(x, y) + \delta = \varepsilon$, and $z \in^\ell \mathbb{B}(x, \varepsilon)$. $\square$

The absence of a true complement for multisets makes it appealing to explore the concept of closedness throughout auxiliary tools as will be described in the sequel.

Definition 8. For a multiset $\mathcal{A} : X \rightarrow \mathbb{N}$, the function $\mathcal{A}^\ominus : X \rightarrow \mathbb{Z}_0^-$ to the set of nonpositive integers is defined by the rule $\mathcal{A}^\ominus(x) := -\mathcal{A}(x)$, and in this case $\mathcal{A}^\ominus$ is called an **antimultiset** over X . For the multipoints $[x]_k$ in X , we adopt the alternative notation $[x]_k \widetilde{\in} \mathcal{A}^\ominus$ when $[x]_k \notin \mathcal{A}$.

Let us clarify that we introduce antimultisets only for notational convenience rather than as a separate subject of interest, and we do not attribute any conceptual meaning to them.

Definition 9. We call an antimultiset $\mathcal{A}^\ominus$ over X , **closed** with respect to a multiset topology $\mathcal{T}$ on X , if $\mathcal{A}$ is an open multiset.

Now, in the context of multimetric spaces, we will examine the relationship between closed sets and convergent sequences.

Definition 10. Let $(X, \mathcal{T})$ be a multiset topological space. A sequence $([x_n]_{k_n})_n$ of multipoints in X is said to be **convergent** to a multipoint $[x]_k$ in X , if for each open multiset $\mathcal{A}$ such that $[x]_k \subset \mathcal{A}$, there exists an $n_0 \in \mathbb{Z}^+$, such that for all $n \geq n_0$, $[x_n]_{k_n} \subset \mathcal{A}$. In this case we write $([x_n]_{k_n})_n \rightarrow [x]_k$.

Proposition 5. If $([x_n]_{k_n})_n$ is a sequence of multipoints on a multiset topological space $(X, \mathcal{T})$ such that $([x_n]_{k_n})_n \rightarrow [x]_k$, then also $([x_n]_{k_n})_n \rightarrow [x]_\ell$ for all $\ell \geq k$.

Proof. If $[x]_\ell \subset \mathcal{A}$ and $\ell \geq k$, then $[x]_k \subset \mathcal{A}$ also holds, and since the convergence hypothesis is already satisfied for such multisets $\mathcal{A}$, this sequence also converges to $[x]_\ell$. $\square$

One of the well-known facts in topology is that the limits of convergent sequences defined in closed sets lie within the set. Below, we provide an analogous result in the context of multiset topology.

Theorem 6. Let $(X, \mathcal{T})$ be a multiset topological space, $\mathcal{A}^\ominus$ be a closed antimultiset, and $([x_n]_{k_n})_n$ be a sequence of multipoints in X , such that $[x_n]_{k_n} \widetilde{\in} \mathcal{A}^\ominus$ for all $n \in \mathbb{Z}^+$ and $([x_n]_{k_n})_n \rightarrow [x]_k$. Then $[x]_k \widetilde{\in} \mathcal{A}^\ominus$.

Proof. Assume that $[x]_k \subset \mathcal{A}$. Since the antimultiset $\mathcal{A}^\ominus$ is closed, the multiset $\mathcal{A}$ is open and by $([x_n]_{k_n})_n \rightarrow [x]_k$, there exists an $n_0 \in \mathbb{Z}^+$, such that for all $n \geq n_0$, $[x_n]_{k_n} \subset \mathcal{A}$. But this contradicts by $[x_n]_{k_n} \widetilde{\in} \mathcal{A}^\ominus$. Therefore $[x]_k \notin \mathcal{A}$, and we have $[x]_k \widetilde{\in} \mathcal{A}^\ominus$. $\square$

For the multiset topologies induced by multimetric spaces, we can state a converse for this theorem.

Theorem 7. Let $(X, \mathcal{d})$ be a multimetric space and $\mathcal{A}^\ominus$ be an antimultiset over X . If for each convergent sequence $([x_n]_{k_n})_n \rightarrow [x]_k$ of multipoints in X , $[x_n]_{k_n} \widetilde{\in} \mathcal{A}^\ominus$ for all $n \in \mathbb{Z}^+$ implies $[x]_k \widetilde{\in} \mathcal{A}^\ominus$, then $\mathcal{A}^\ominus$ is closed.

Proof. Assume the contrary that $\mathcal{A}^\ominus$ is not closed. Then $\mathcal{A}$ is not open. In this case, there is an $x \in^k \mathcal{A}$, such that for all $\varepsilon > 0$, $\mathbb{B}(x, \varepsilon) \not\subset^k \mathcal{A}$, so that there exists an $x_\varepsilon \in^{\ell_\varepsilon} \mathbb{B}(x, \varepsilon)$ where $\ell_\varepsilon \leq k$, but $x_\varepsilon \notin^{\ell_\varepsilon} \mathcal{A}$, that is $[x_\varepsilon]_{\ell_\varepsilon} \not\subset \mathcal{A}$, or equivalently $[x_\varepsilon]_{\ell_\varepsilon} \widetilde{\in} \mathcal{A}^\ominus$. In particular, for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{Z}^+$, we have a sequence $([x_{1/n}]_{\ell_{1/n}})_n$ of multipoints. We claim that this sequence converges to $[x]_k$.

Let $\mathcal{G}$ be an open multiset such that $[x]_k \subset \mathcal{G}$, that is $x \in^k \mathcal{G}$. Then there exists a $\delta > 0$ such that $\mathbb{B}(x, \delta) \subset^k \mathcal{G}$. Pick an $n_0 \in \mathbb{Z}^+$ such that $\frac{1}{n_0} < \delta$. Then by the selection of the terms of the sequence and Proposition 4 2, we have

$$x_{1/n} \in^{\ell_{1/n}} \mathbb{B}(x, \frac{1}{n}) \subset \mathbb{B}(x, \delta) \subset \mathcal{G}$$

for all $n \geq n_0$, so that $[x_{1/n}]_{\ell_{1/n}} \subset \mathcal{G}$, which means that $([x_{1/n}]_{\ell_{1/n}})_n \rightarrow [x]_k$. This implies, by the hypothesis, that $[x]_k \widetilde{\in} \mathcal{A}^\ominus$ since $[x_{1/n}]_{\ell_{1/n}} \widetilde{\in} \mathcal{A}^\ominus$ for all $n \in \mathbb{Z}^+$. On the other hand, $[x]_k \widetilde{\in} \mathcal{A}^\ominus$ is equivalent to $[x]_k \notin \mathcal{A}$ and this contradicts by $x \in^k \mathcal{A}$. Hence $\mathcal{A}^\ominus$ has to be closed. $\square$

5.3 Multitopology

Through this subsection, we suppose, without further notification, that I is an index set, $A, A_i, B \subseteq X$ and $k, k_i, \ell \in \mathbb{Z}^+$ for all $i \in I$.

Definition 11. A **multitopology** on a set X , is a collection $\mathcal{T} = \{\tau_k : k \in \mathbb{Z}^+\}$ of families of subsets of X , such that

- a) $X \in \tau_1$ and $\emptyset \in \tau_k$ for all $k \in \mathbb{Z}^+$.
- b) If $k_i \in \mathbb{Z}^+$, $A_i \in \tau_{k_i}$ for all $i \in I$ and $k = \min\{k_i : i \in I\}$, then $\bigcup_{i \in I} A_i \in \tau_k$.
- c) If $k, \ell \in \mathbb{Z}^+$, $A \in \tau_k$ and $B \in \tau_\ell$, then $A \cap B \in \tau_{\min(k, \ell)}$.

In this case, the pair $(X, \mathcal{T})$ is called a **multitopological space**, and the sets in τ_k are called **k -open**.

Similar to the situation in multimetric spaces, multitopological spaces yield a topology, namely if $(X, \mathcal{T})$ is a multitopological space, then (X, τ_1) is a topological space. This is called the **topological support** of $(X, \mathcal{T})$, and in this case, we write $|(X, \mathcal{T})| := (X, \tau_1)$. However, τ_2, τ_3 and beyond do not have to be topologies on X , as they do not necessarily contain X .

The families τ_k in the definition of a multitopology are not completely independent from one another. The following proposition expresses an important relationship between these.

Proposition 6. If $(X, \mathcal{T})$ is a multitopological space and $k \leq \ell$, then each ℓ -open subset of X is also k -open.

Proof. If $A \in \tau_\ell$, then by $\emptyset \in \tau_k$ we have $A = A \cup \emptyset \in \tau_{\min(\ell, k)} = \tau_k$. □

For a given multitopological space $(X, \mathcal{T})$, we define the sets $X_{\bar{k}}$ by

$$X_{\bar{k}} := \bigcup_{A \in \tau_k} A$$

for each $k \in \mathbb{Z}^+$. Then one always has $X_{\bar{1}} = X$ and $X_{\bar{1}} \supset X_{\bar{2}} \supset X_{\bar{3}} \supset X_{\bar{4}} \supset \dots$.

Definition 12. In a multitopological space $(X, \mathcal{T})$, a subset $A \subset X$ in the form $A = X_{\bar{k}} \setminus G$ where G is k -open, is called **k -closed**.

Even though we are dealing with a more partial complement rather than one over X , from the definition of the sets $X_{\bar{k}}$, it is clear that every k -open set is a subset of $X_{\bar{k}}$. Therefore, if G is k -open, then $G \subset X_{\bar{k}}$, and in this case we have a complete duality between k -open and k -closed sets via the equality $G = X_{\bar{k}} \setminus (X_{\bar{k}} \setminus G)$.

To clarify the relationship between multitopological spaces and multimetric spaces, we present the following definition, and immediately afterward, a related result.

Definition 13. In a multimetric space $(X, \mathcal{d})$, a subset $A \subseteq X$ is called **k -open** if $d_k(a, a) = 0$ and there exists an $\varepsilon > 0$ such that $\mathbb{B}(a, \varepsilon) \subseteq A$, for each $a \in A$.

Theorem 8. Let τ_k be the family of all k -open subsets in a multimetric space $(X, \mathcal{d})$. Then the families τ_k ($k \in \mathbb{Z}^+$) define a multitopology on X .

Proof. $X \in \tau_1$, since $d_1(x, x) = 0$ and $\mathbb{B}(a, \varepsilon) \subseteq X$ for all $x \in X$. On the other hand, the statement that $\emptyset \in \tau_k$ for all k , is vacuously true.

Let $k_i \in \mathbb{Z}^+$, $A_i \in \tau_{k_i}$ for all $i \in I$, and $a \in \bigcup_{i \in I} A_i$. Then $d_{k_i}(a, a) = 0$ and there are $\varepsilon_i > 0$ for all $i \in I$, such that $\mathbb{B}(a, \varepsilon_i) \subseteq A_i$. If we say $k = \min\{k_i : i \in I\}$, then also $d_k(a, a) = 0$ and $\mathbb{B}(a, \varepsilon_i) \subseteq A_i$ for

all $i \in I$, by $k \leq k_i$. Also by $A_{i_0} \subseteq \bigcup_{i \in I} A_i \in \tau_k$, for any fixed $i_0 \in I$, we have $\mathbb{B}(a, \varepsilon_{i_0}) \subseteq \bigcup_{i \in I} A_i$, which proves $\bigcup_{i \in I} A_i \in \tau_k$.

Finally, suppose that $A \in \tau_k$, $B \in \tau_\ell$, and $c \in A \cap B$. Then there are $\delta, \varepsilon > 0$ such that $d_k(c, c) = 0$, $\mathbb{B}(c, \delta) \subseteq A$, $d_\ell(c, c) = 0$ and $\mathbb{B}(c, \varepsilon) \subseteq B$. In this case, $d_{\min(k, \ell)}(c, c) = 0$ and moreover we have $\mathbb{B}(c, \delta) \cap \mathbb{B}(c, \varepsilon) = \mathbb{B}(c, \min(\delta, \varepsilon)) \subseteq A \cap B$ by Proposition 4 3, resulting in $A \cap B \in \tau_{\min(k, \ell)}$. $\square$

Proposition 7. *Let $(X, \mathcal{d})$ be a multimetric space and $x \in X$. Then the set $|\mathbb{B}(x, \varepsilon)|$ is k -open for all $\varepsilon > 0$, if $d_k(x, x) = 0$.*

Proof. Clearly $\mathbb{B}(x, \varepsilon) \subseteq |\mathbb{B}(x, \varepsilon)|$ and we have $d_k(x, x) = 0$ by the hypothesis. Therefore the set $|\mathbb{B}(x, \varepsilon)|$ is k -open. $\square$

Definition 14. *In a multitopological space, a sequence (x_n) is called **k -convergent** to a point x , if for each k -open set G with $x \in G$, there is an $n_0 \in \mathbb{Z}^+$, such that $x_n \in G$ for all $n \geq n_0$. In this case we write $(x_n) \xrightarrow{k} x$.*

In the context of Proposition 6, while k -openness is a weaker property than ℓ -openness when $k \leq \ell$, the situation reverses in terms of convergence, and k -convergence becomes a stronger property than ℓ -convergence.

Proposition 8. *If $(X, \mathcal{T})$ is a multitopological space, (x_n) is a sequence on X and $k \leq \ell$, then $(x_n) \xrightarrow{k} x$ implies $(x_n) \xrightarrow{\ell} x$.*

Proof. If $(x_n) \xrightarrow{k} x$, then for each k -open set G , $x \in G$ implies the existence of an $n_0 \in \mathbb{Z}^+$, such that $x_n \in G$ for all $n \geq n_0$. By $k \leq \ell$, each ℓ -open set is also k -open. Hence the corresponding condition holds also for ℓ -open sets, so that $(x_n) \xrightarrow{\ell} x$. $\square$

Now, we present a result that establishes a relationship between k -closedness and k -convergence in multitopological spaces.

Theorem 9. *Let $(X, \mathcal{T})$ be a multitopological space, A be a k -closed set, and (x_n) be a sequence in A . If $(x_n) \xrightarrow{k} x \in X_{\bar{k}}$, then $x \in A$.*

Proof. We assume the contrary. Let $x \notin A$. Since A is k -closed, it is of the form $A = X_{\bar{k}} \setminus G$, where G is a k -open set. In this case also $G \subseteq X_{\bar{k}}$ and $G = X_{\bar{k}} \setminus A$. Since $x \in X_{\bar{k}}$, we have $x \in G$ by the assumption that $x \notin A$. Then G is a k -open set including the limit x , and therefore $x_n \in G$ except for a finite number of terms. But this is a contradiction since (x_n) is a sequence in A , and $A \cap G = \emptyset$. This forces $x \in A$. $\square$

Finally, we show that a converse for the previous result holds specifically for the multitopologies of multimetric spaces.

Theorem 10. *Let $(X, \mathcal{d})$ be a multimetric space and $A \subseteq X_{\bar{k}}$. If $(x_n) \xrightarrow{k} x \in X_{\bar{k}}$ implies $x \in A$ for each sequence (x_n) in A , then A is k -closed.*

Proof. Say $G = X_{\bar{k}} \setminus A$. Assume that A is not k -closed. Then G is not k -open. Since X_k is an union of k -open subsets by its definition, $x \in X_{\bar{k}}$ gives $d_k(x, x) = 0$. In particular, by $G \subseteq X_k$, $d_k(x, x) = 0$ for all $x \in G$. Since G is not k -open, in this case, there must be an $x \in G$ such that $\mathbb{B}(x, \varepsilon) \not\subseteq G$ for all $\varepsilon > 0$, that is $|\mathbb{B}(x, \varepsilon)| \not\subseteq G$.

For each $n \in \mathbb{Z}^+$, pick an $x_n \in |\mathbb{B}(x, \frac{1}{n})|$ such that $x_n \notin G$. By Proposition 5.3, the sets $|\mathbb{B}(x, \frac{1}{n})|$ are k -open, so that $|\mathbb{B}(x, \frac{1}{n})| \subseteq X_{\bar{k}}$. Hence $x_n \in X_{\bar{k}} \setminus G = A$, and (x_n) is a sequence in A .

Let H be a k -open set such that $x \in H$. Then there is an $\varepsilon > 0$ such that $\mathbb{B}(x, \varepsilon) \subseteq H$ for all $\varepsilon > 0$. In this case $|\mathbb{B}(x, \frac{1}{n})| \subseteq H$, if $\frac{1}{n} < \varepsilon$, $n \in \mathbb{Z}^+$. Choosing $n_0 \in \mathbb{Z}^+$ such that $\frac{1}{n_0} < \varepsilon$ gives $x_n \in H$ for all $n \geq n_0$. Thus (x_n) k -converges to x .

Consequently, (x_n) is a sequence in A and $(x_n) \xrightarrow{k} x \in X_{\bar{k}}$. So $x \in A$ by the hypothesis, and this contradicts by $x \in G = X_{\bar{k}} \setminus A$. Therefore A has to be k -closed. $\square$

6 Conclusion

This research, in which various concepts such as multinumber, multimetric, multinorm, and multitopology are introduced and examined, serves as an introduction to the field, laying the foundational groundwork and preparing the way for future research and applications. The study of these structures, from the perspective of mathematical and particularly topological concepts, their logical implications, statistical applications, and combinatorial aspects, leaves many potential early questions that will be awaiting answers.

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References

- [1] Akçay, D. and Alaca, C. (2024). Coupled fixed point theorems in dislocated quasi-metric spaces. *Journal of Advanced Studies in Topology*, 4(1):66.

- [2] Al Tahan, M., Hosková-Mayerová, S., and Davvaz, B. (2020). Fuzzy multi-polygroups. *Journal of Intelligent & Fuzzy Systems*, 38(2):2337–2345.
- [3] Al Tahan, M., Hosková-Mayerová, S., and Davvaz, B. (2021). An approach to fuzzy multi-ideals of near rings. *Journal of Intelligent & Fuzzy Systems*, 41(6):6233–6243.
- [4] Alhazov, A., Freund, R., Ivanov, S., and Verlan, S. (2022). Tissue p systems with vesicles of multisets. *International Journal of Foundations of Computer Science*, 33(03N04):179–202.
- [5] Ali, N., Siddiqui, H. M. A., Qureshi, M. I., Abdallah, S. A. O., Almahri, A., Asad, J., and Akgül, A. (2024). Exploring ring structures: multiset dimension analysis in compressed zero-divisor graphs. *Symmetry*, 16(7).
- [6] Alias, S., Mohamad, D., and Shuib, A. (2017). Rough neutrosophic multisets. *Neutrosophic Sets and Systems*, 16:80–88.
- [7] Aman, B. and Ciobanu, G. (2021). Knowledge dynamics and behavioural equivalences in multi-agent systems. *Mathematics*, 9(22).
- [8] Bekmetjev, A., Brightwell, G., Czygrinow, A., and Hurlbert, G. (2003). Thresholds for families of multisets, with an application to graph pebbling. *Discrete Mathematics*, 269(1–3):21–34.
- [9] Bernardini, F., Brijder, R., Cavaliere, M., Franco, G., Hoogeboom, H. J., and Rozenberg, G. (2011). On aggregation in multiset-based self-assembly of graphs. *Natural Computing*, 10(1):17–38.
- [10] Bernardini, F., Brijder, R., Rozenberg, G., and Zandron, C. (2007). Multiset-based self-assembly of graphs. *Fundamenta Informaticae*, 75(1–4):49–75.
- [11] Cão, L. Z., Féng, Y. M., and Sangodapo, T. O. (2024). Picture fuzzy multisets. *Italian Journal of Pure and Applied Mathematics*, 51:64–76.
- [12] Cavaliere, M., Freund, R., Oswald, M., and Sburlan, D. (2007). Multiset random context grammars, checkers, and transducers. *Theoretical Computer Science*, 372(2–3):136–151.
- [13] Çetin, S. and Gürdal, U. (2023). Characterization of bipolar ultrametric spaces and fixed point theorems. *Hacettepe Journal of Mathematics and Statistics*, 52(1):185–196.
- [14] Çetin, S., Kişi, Ö., and Gürdal, M. (2025). Exploration of novel convergence concepts for sequences in octonion-valued metric spaces. *Advances in Mathematical Sciences and Applications*, 34(1):271–300.
- [15] Demir, N. and Gümüş, H. (2023). Ideal convergence of multiset sequences. *Filomat*, 37(30):10199–10207.
- [16] Dhingra, M., Dubey, M. K., and Kavikumar, J. (2023). L-Fuzzy Multiset Recognizer: Determinization and Minimization. *IEEE Transactions on Emerging Topics in Computational Intelligence*, 7(4):1163–1176.
- [17] Ege, Ö. and Karaca, İ. (2018). Complex valued dislocated metric spaces. *Korean Journal of Mathe-*

- matics*, 26(4):809–822.
- [18] Felisiak, P. A., Qín, K. Y., and Lǐ, G. (2020). Generalized multiset theory. *Fuzzy Sets and Systems*, 380:104–130.
 - [19] Féng, Y. M. and Onasanya, B. O. (2019). Multigroups and multicosests. *Italian Journal of Pure and Applied Mathematics*, 41:251–261.
 - [20] García-Pacheco, F. J. and Kama, R. (2024). On modulus statistical convergence in partial metric spaces. *Axioms*, 13(6):388.
 - [21] Girish, K. P. and John, S. J. (2012). Multiset topologies induced by multiset relations. *Information Sciences*, 188:298–313.
 - [22] Guān, H., Khan, A., Akhter, S., and Hameed, S. (2023). Spectral characterization of graphs with respect to the anti-reciprocal eigenvalue property. *Symmetry*, 15(6):1240.
 - [23] Gürdal, M. (2009). Description of extended eigenvalues and extended eigenvectors of integration operators on the wiener algebra. *Expositiones Mathematicae*, 27(2):153–160.
 - [24] Gürdal, M., Garayev, M. T., and Saltan, S. (2015). Some concrete operators and their properties. *Turkish Journal of Mathematics*, 39(6):970–989.
 - [25] Gürdal, U., Oğur, Ö., and Çetin, S. (2025). Unima on $[0,1]$. *Fuzzy Sets and Systems*, page 109439.
 - [26] Gürdal, U., Yapalı, R., and Mutlu, A. (2024). Generalized Kuratowski closure operators in the bipolar metric setting. *Proceedings of International Mathematical Sciences*, 6(1):1–17.
 - [27] Hoque, M. M., Bhattacharya, B., and Tripathy, B. C. (2024). On mixed multiset ideal topological spaces. *Filomat*, 38(15):5517–5526.
 - [28] Hosková-Mayerová, S., Al Tahan, M., and Davvaz, B. (2020). Fuzzy multi-hypergroups. *Mathematics*, 8(2):244.
 - [29] Hosková-Mayerová, S. and Onasanya, B. O. (2019). Results on functions on Dedekind multisets. *Symmetry*, 11(9):1125.
 - [30] Ilamathi, V. S. A., Vimala, J., and Davvaz, B. (2019). Multiset filters of residuated lattices and its application in medical diagnosis. *Journal of Intelligent & Fuzzy Systems*, 36(3):2297–2305.
 - [31] Kandil, A., El-Sheikh, S. A., Hosny, M., and Raafat, M. (2020). Hesitant fuzzy soft multisets and their applications in decision-making problems. *Soft Computing*, 24(6):4223–4232.
 - [32] Karaev, M. T., Gürdal, M., and Huban, M. B. (2016). Reproducing kernels, enliş algebras and some applications. *Studia Mathematica*, 232(2):113–141.
 - [33] Karaev, M. T., Gürdal, M., and Saltan, S. (2011). Some applications of Banach algebra techniques. *Mathematische Nachrichten*, 84(13):1678–1689.

- [34] Kharal, A., Alshehri, M. H., Bin Turki, N., and Duraihem, F. Z. (2021). Generalized mapping for multiset rewriting systems. *Soft Computing*, 25(17):11439–11448.
- [35] Kovačević, M. (2025). Vector multispaces and multispace codes. *Applied Mathematics and Computation*, 486:129041.
- [36] Law, Y. C., Lee, J. H. M., Walsh, T., and Woo, M. H. C. (2013). Multiset variable representations and constraint propagation. *Constraints*, 18(3):307–343.
- [37] Lǐ, J. Y. and Yú, X. (2021). A new sieve for restricted multiset counting. *Discrete Mathematics*, 344(8):112487.
- [38] Liào, J. Q., Lǚ, Z. Q., Cáo, M. Y., and Lù, M. (2024). Erdős-ko-rado theorem for bounded multisets. *Journal of Combinatorial Theory Series A*, 206:105888.
- [39] Miyamoto, S. (2000). Multisets and fuzzy multisets. In *Soft Computing and Human-Centered Machines*, pages 9–33. Springer Japan, Tokyo.
- [40] Molaei, M. R. and Hoseini, A. (1999). Generalized groups. *Bull. Inst. Polit. Di. Iase Fasc*, 3(4):21–24.
- [41] Murai, T., Miyamoto, S., Inuiguchi, M., Kudo, Y., and Akama, S. (2015). Fuzzy multisets in granular hierarchical structures generated from free monoids. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 19(1):41–48.
- [42] Nabiev, A. A., Savaş, E., and Gürdal, M. (2019). Statistically localized sequences in metric spaces. *Journal of Applied Analysis and Computation*, 9(2):739–746.
- [43] Onar, S., Ersoy, B. A., Davvaz, B., and Hila, K. (2022). Intuitionistic fuzzy multi-polygroups. *Computational & Applied Mathematics*, 41(1):12.
- [44] Öneş, O., Alkan, M., and Uc, M. (2020). On submodules of modules over group rings. *Filomat*, 34(2):575–582.
- [45] Özkan, K. (2023). Common fixed point results for Suzuki type contractions on partial metric spaces with an application. *Cumhuriyet Science Journal*, 44(1):148–159.
- [46] Paul, J. and John, S. J. (2016). Type 2 fuzzy multisets and its applications in decision making problems. *Journal of Intelligent & Fuzzy Systems*, 30(1):359–369.
- [47] Ramaswamy, R., Mani, G., Gnanaprakasam, A. J., Abdelnaby, O. A. A., Stojiljković, V., Radojević, S., and Radenović, S. (2022). Fixed points on covariant and contravariant maps with an application. *Mathematics*, 10(22):4385.
- [48] Sòng, Y., Gè, X., and Lǐ, Z. W. (2020). A note on rough multisets. *Journal of Intelligent & Fuzzy Systems*, 39(3):3899–3906.
- [49] Suma, P. and John, S. J. (2023). Multiset group and its generalization to (a, b) -multiset group. *Italian Journal of Pure and Applied Mathematics*, 50:602–617.

- [50] Tripathy, B. C. and Das, R. (2023). Multiset mixed topological space. *Transactions of A. Razmadze Mathematical Institute*, 177(1):119–123.
- [51] Uc, M. and Alkan, M. (2023). On modules over g -sets. *Journal of Mathematics and Statistics Studies*, 4(4):47–55.
- [52] Uc, M., Öneş, O., and Alkan, M. (2016). On modules over groups. *Filomat*, 30(4):1021–1027.
- [53] Uluçay, V. and Şahin, M. (2019). Neutrosophic multigroups and applications. *Mathematics*, 7(1):95.
- [54] Yager, R. R. (1986). On the theory of bags. *International Journal of General Systems*, 13(1):23–37.
- [55] Zakaria, A., John, S. J., and El-Sheikh, S. A. (2016). Generalized rough multiset via multiset ideals. *Journal of Intelligent & Fuzzy Systems*, 30(3):1791–1802.